Problem Solving_sec04

7 Aralık 2020 Pazartesi 22:42

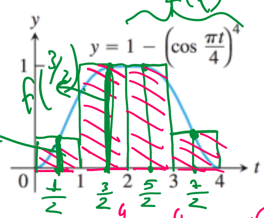
Average Value of a Function

In Exercises 15–18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad \text{on} \quad [0, 4]$$

Sum of area of 4-rectangle $= 1 \cdot f\left(\frac{1}{2}\right) + 1 \cdot f\left(\frac{3}{2}\right) + 1 \cdot f\left(\frac{5}{2}\right) + 1 \cdot f\left(\frac{7}{2}\right)$

$= 1 - \cos^4\left(\frac{\pi}{8}\right) + 1 - \cos^4\left(\frac{3\pi}{8}\right) + 1 - \cos^4\left(\frac{5\pi}{8}\right) + 1 - \cos^4\left(\frac{7\pi}{8}\right)$



$$Av(f) = \frac{\int_a^b f(t)\,dt}{b-a} = \frac{\int_0^4 1 - \cos^4\left(\frac{\pi t}{4}\right) dt}{4-0} \approx \frac{\text{Approximation for the definite integral}}{4}$$

21. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) **b.** 8 (octagon) **c.** 16

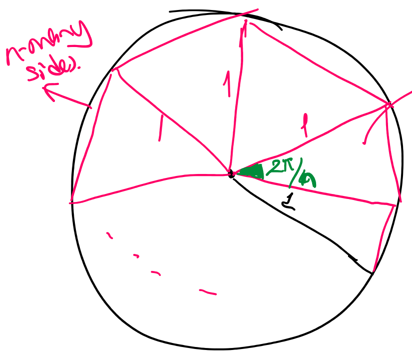
d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

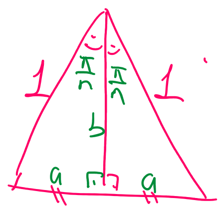
22. (*Continuation of Exercise 21.*)

a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

c. Repeat the computations in parts (a) and (b) for a circle of radius r.





$\sin\left(\frac{\pi}{n}\right) = \frac{a}{1} \Rightarrow a = \sin\left(\frac{\pi}{n}\right)$

$\cos\left(\frac{\pi}{n}\right) = \frac{b}{1} \Rightarrow b = \cos\left(\frac{\pi}{n}\right)$

$A_{\text{triangle}} = \frac{1}{\cancel{2}} \cancel{2} a \cdot b = \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$

Area of polygon $= n \cdot \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$

Recall

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Area of circle $= \lim_{n \to \infty}$ area of polygon $= \lim_{n \to \infty} n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)$

$= \lim_{n \to \infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}\pi} \pi = \pi$

$\pi(1^2) = \pi$

Evaluate the sums

25. $\sum_{k=1}^{5} k(3k + 5)$ **26.** $\sum_{k=1}^{7} k(2k + 1)$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

Recall

$\sum^{n}$ ...

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ 28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

29. a. $\sum_{k=1}^{7} 3$ b. $\sum_{k=1}^{500} 7$ c. $\sum_{k=3}^{264} 10$

30. a. $\sum_{k=9}^{36} k$ b. $\sum_{k=3}^{17} k^2$ c. $\sum_{k=18}^{71} k(k-1)$

31. a. $\sum_{k=1}^{n} 4$ b. $\sum_{k=1}^{n} c$ c. $\sum_{k=1}^{n} (k-1)$

32. a. $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ b. $\sum_{k=1}^{n} \frac{c}{n}$ c. $\sum_{k=1}^{n} \frac{k}{n^2}$

Recall

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \dots + a_n$$

- $\sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

(26)

$$\sum_{k=1}^{7} k\cdot(2k+1) = \sum_{k=1}^{7} 2k^2 + k$$

$$= \sum_{k=1}^{7} 2k^2 + \sum_{k=1}^{7} k$$

$$= 2\underbrace{\sum_{k=1}^{7} k^2}_{\frac{7\cdot 8\cdot 15}{6}} + \underbrace{\sum_{k=1}^{7} k}_{\frac{7\cdot 8}{2}} =$$

$$= \frac{7\cdot 8\cdot 15}{3} + 7\cdot 4$$

(29) c)

$$\sum_{k=3}^{264} 10 = \underbrace{10 + 10 + \dots + 10}_{\text{number of terms: } 264-3+1 = 262} = 262\cdot 10 = 2620$$

k=3 k=4 k=264

(32) a)

$$\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right) = \left(\frac{1}{n} + 2n\right) + \left(\frac{1}{n} + 2n\right) + \dots + \left(\frac{1}{n} + 2n\right)$$

k=1 k=2 k=n

$$= n\cdot\left(\frac{1}{n} + 2n\right) = 1 + 2n^2$$

number of terms

c)

$$\sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \underbrace{\sum_{k=1}^{n} k}_{\frac{n(n+1)}{2}} = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2n}$$

Limits of Riemann Sums

For the functions in Exercises 39–46, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

39. $f(x) = 1 - x^2$ over the interval $[0, 1]$.

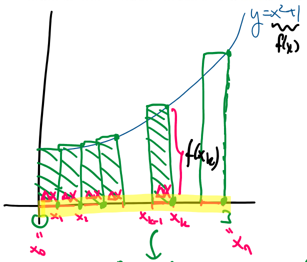
40. $f(x) = 2x$ over the interval $[0, 3]$.

41. $f(x) = x^2 + 1$ over the interval $[0, 3]$.

42. $f(x) = 3x^2$ over the interval $[0, 1]$.

43. $f(x) = x + x^2$ over the interval $[0, 1]$.

44. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.



Area of the k-th rectangle $= A_k = \underbrace{\Delta x}_{\text{base}} \cdot \underbrace{f(x_k)}_{\text{height}}$

Sum of the areas of rectangles $= S_n = \overbrace{\sum_{k=1}^{n} \Delta x \cdot f(x_k)}^{\text{Riemann Sum}}$

$y = f(x) \geq 0$

Exact area under the curve $= \int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \Delta x\, f(x_k)$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(x_k)\right)}$$

where $\boxed{\Delta x = \frac{3}{n}}$, $\boxed{x_k = 0 + k\cdot\Delta x = \frac{3k}{n}}$

$$S_n = \sum_{k=1}^{n} \Delta x\, f(x_k)$$
$$= \sum_{k=1}^{n} \frac{3}{n} \cdot f\left(\frac{3k}{n}\right)$$
$$= \sum_{k=1}^{n} \frac{3}{n} \cdot \left[\left(\frac{3k}{n}\right)^2 + 1\right] \Rightarrow$$
$$= \sum_{k=1}^{n} \frac{3}{n} \cdot \left(\frac{9k^2}{n^2} + 1\right)$$
$$= \sum_{k=1}^{n} \frac{27k^2}{n^3} + \sum_{k=1}^{n} \frac{3}{n}$$
$$= \frac{27}{n^3} \underbrace{\sum_{k=1}^{n} k^2}_{\frac{n(n+1)(2n+1)}{6}} + \underbrace{\sum_{k=1}^{n} \frac{3}{n}}_{n \cdot \frac{3}{n}}$$
$$= \frac{\cancel{27}^{9}}{n^{\cancel{3}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_2} + 3$$
$$S_n = \frac{9(n+1)(2n+1)}{2n^2} + 3$$

$$\lim_{n\to\infty} \sum \left(\frac{3}{n}\Big(\big(\tfrac{3k}{n}\big)^2 + 1\Big)\right) = \int_{a=0}^{b=3} f(x)\,dx = \left(\frac{x^3}{3} + \frac{x}{1}\right)\Bigg|_0^3 = 12$$

$f(x_k) = x_k^2 + 1$, $\quad f(x) = x^2+1$, $\quad \frac{3k}{n} = x_k$, $\quad \Delta x = \frac{3-0}{n} = \frac{3}{n}$

Exact area under the curve $y = x^2+1$ $= \int_0^3 \underbrace{(x^2+1)}_{f(x)}\,dx = \lim_{n\to\infty}\left(\underbrace{\sum_{k=1}^{n} \Delta x\, f(x_k)}_{S_n}\right)$

$$= \lim_{n\to\infty}\left(\underbrace{\frac{9(n+1)(2n+1)}{2n^2}}_{\frac{9\cdot 2}{2}} + 3\right)$$

$$= \frac{18}{2} + 3 = 12$$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

16. $\int_{1/2}^{3/2} (-2x + 4)\, dx$

17. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

18. $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$

19. $\int_{-2}^{1} |x|\, dx$

20. $\int_{-1}^{1} (1 - |x|)\, dx$

21. $\int_{-1}^{1} (2 - |x|)\, dx$

22. $\int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^{4}\left(\frac{x}{2}+3\right)dx$ **16.** $\int_{1/2}^{3/2}(-2x+4)\,dx$

17. $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ **18.** $\int_{-4}^{0}\sqrt{16-x^2}\,dx$

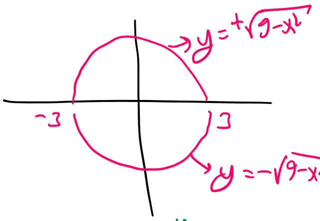
19. $\int_{-2}^{1}|x|\,dx$ **20.** $\int_{-1}^{1}(1-|x|)\,dx$

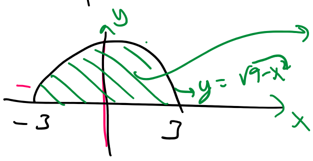
21. $\int_{-1}^{1}(2-|x|)\,dx$ **22.** $\int_{-1}^{1}\left(1+\sqrt{1-x^2}\right)dx$

(17) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ = area under the curve $y=\sqrt{9-x^2}$

$x^2+y^2=\underbrace{9}_{r^2}$ → circle of radius = 3

$y=\pm\sqrt{9-x^2}$

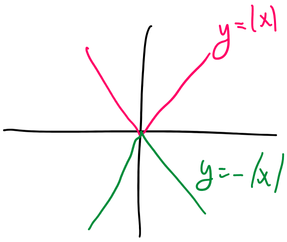




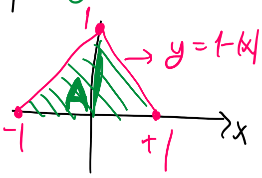
Area $= \int_{-3}^{+3}\sqrt{9-x^2} = \frac{9\pi}{2}$

$\frac{\pi(3^2)}{2}=\frac{9\pi}{2}$

(20)



$\int_{-1}^{+1}\underbrace{1-|x|}_{y}\,dx = A = \frac{1}{2}\cdot 2\cdot 1 = 1$



→ $y=1-|x| \Rightarrow 0 = 1-|x|$

$|x|=1 \Rightarrow x=+1,\ x=-1$

Average Value of a Continuous Function Revisited

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x)=x^2-1$ on $[0,\sqrt{3}]$

56. $f(x)=-\frac{x^2}{2}$ on $[0, 3]$ **57.** $f(x)=-3x^2-1$ on $[0, 1]$

58. $f(x)=3x^2-3$ on $[0, 1]$

59. $f(t)=(t-1)^2$ on $[0, 3]$

60. $f(t)=t^2-t$ on $[-2, 1]$

61. $g(x)=|x|-1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x)=-|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

(60) $\text{Av}(f)=\dfrac{\int_{-2}^{1}t^2-t\,dt}{1-(-2)}=\dfrac{\left(\frac{t^3}{3}-\frac{t^2}{2}\right)\Big|_{-2}^{+1}}{3}$

$t=1$, $t=-2$

$\dfrac{\left(\frac{1}{3}-\frac{1}{2}\right)-\left(\frac{(-2)^3}{3}-\frac{(-2)^2}{2}\right)}{3}$

$=\frac{3}{2}$

Recall antiderivative — integral

$x^n \rightarrow \frac{x^{n+1}}{n+1}$

derivative

$\int_a^b x^n\,dx=\frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1}-\frac{a^{n+1}}{n+1}$

$= \frac{2}{2}$

$$\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

Ex) What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx? = \left(\frac{x^5}{5} - \frac{2x^3}{3}\right)\Big|_{x=a}^{x=b}$$

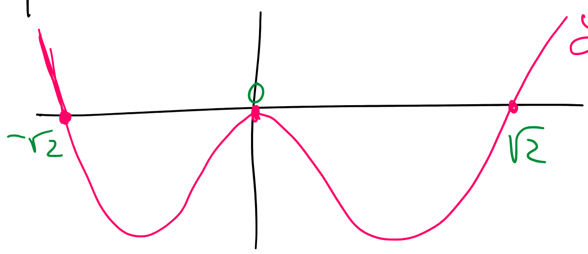
$$= \underbrace{\left(\frac{b^5}{5} - \frac{2b^3}{3}\right) - \left(\frac{a^5}{5} - \frac{2a^3}{3}\right)}_{f(a,b)}$$

$f(a,b)$ ⇓ a fnc of two variable ⇓ and in Calc1 course we can not optimize it !

$y = x^4 - 2x^2$

$y' = 4x^3 - 4x = 4x(x^2+1)$ $\begin{cases} x=0 \\ x=\pm 1 \end{cases}$

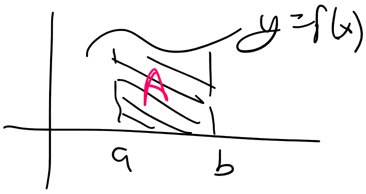
		-1		0		1	
y'	−	0	+	0	−	0	+



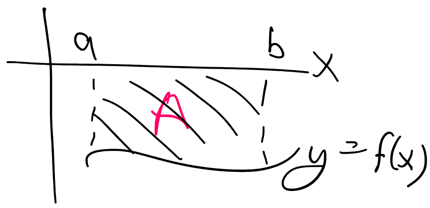
$y = x^4 - 2x^2$ ⇓ $0 = x^2(x^2-2)$

$x = 0$ $\quad$ $x = \pm\sqrt{2}$

Recalls

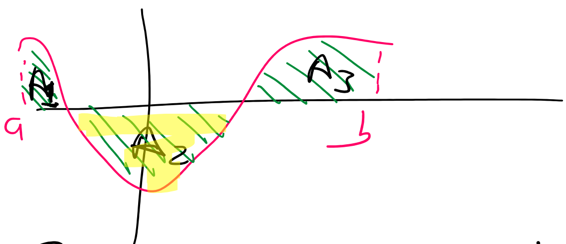
- 

$\int_a^b f(x)\,dx = A$

- 

$f(x) < 0$ ⇓ $\int_a^b f(x)\,dx < 0$

$\int_a^b f(x)\,dx = -A$

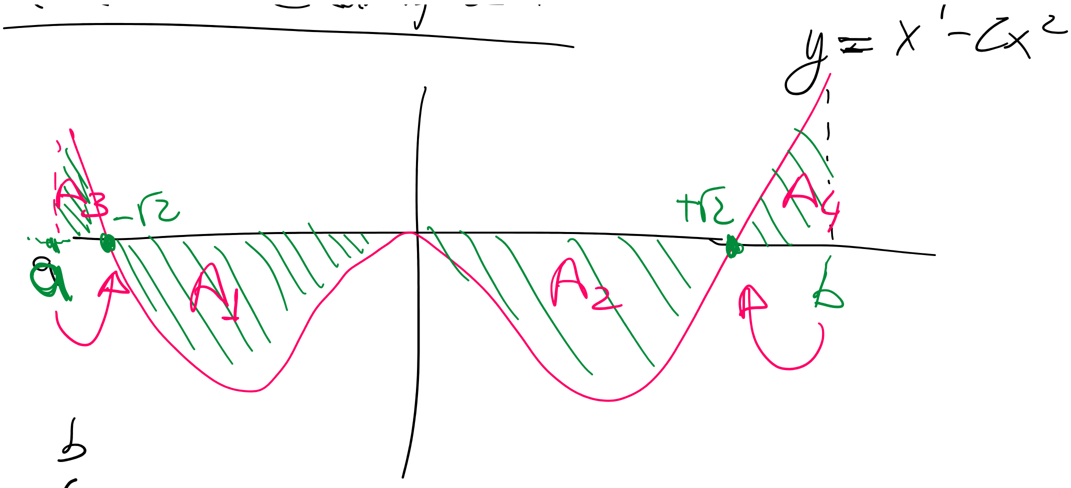


$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

Back to example:

$y = x^4 - 2x^2$

y'



$$\int_a^b x^4 - 2x^2 = -A_1 - A_2 + \underbrace{A_3 + A_4}$$

We want to avoid positive contribution

$\Downarrow$

We need to choose

$$\boxed{\begin{aligned} a &= -\sqrt{2} \\ b &= +\sqrt{2} \end{aligned}} \Rightarrow$$

only then integral gets its smallest possible value.

$\rightarrow$ minimizes the result of definite integral.