Problem Solving_Template

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Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

5. $f'(x) = (x - 1)e^{-x}$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

Identifying Extrema

In Exercises 15–44:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

31. $f(x) = x - 6\sqrt{x - 1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$

38. $g(x) = x^{2/3}(x + 5)$

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2 + 4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$

$$f''(x) = \frac{4}{x^3}$$

3. *Behavior at critical points.*

4. *Increasing and decreasing.*

5. *Inflection points.*

6. *Asymptotes.*

7. The graph of f is sketched in Figure 4.32.

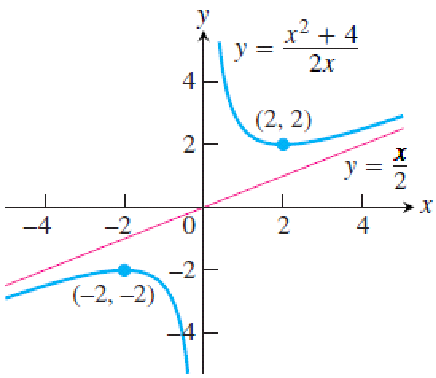

FIGURE 4.32 The graph of $y = \dfrac{x^2 + 4}{2x}$

(Example 9).

EXAMPLE 10 Sketch the graph of $f(x) = e^{2/x}$.

Solution The domain of f is $(-\infty, 0) \cup (0, \infty)$ and there are no symmetries about either axis or the origin. The derivatives of f are

$$f'(x) = e^{2/x}\left(-\frac{2}{x^2}\right) = -\frac{2e^{2/x}}{x^2}$$

and

$$f''(x) = \frac{x^2(2e^{2/x})(-2/x^2) - 2e^{2/x}(2x)}{x^4} = \frac{4e^{2/x}(1 + x)}{x^4}.$$

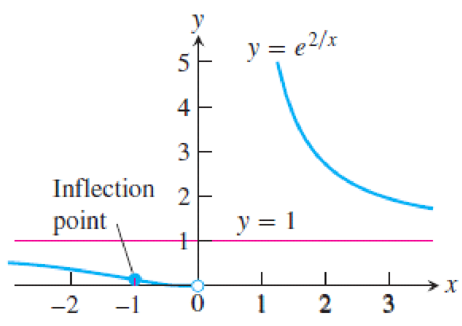

FIGURE 4.33 The graph of $y = e^{2/x}$ has a point of inflection at $(-1, e^{-2})$. The line $y = 1$ is a horizontal asymptote and $x = 0$ is a vertical asymptote (Example 10).

112. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

4.5 Indeterminate Forms and L'Hôpital's Rule

Determinate-Indeterminate Forms Table	
Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty) \cdot (\infty) = \infty$
1^{∞}	
∞^0	
Use L'Hôpital's Rule	Do *Not* Use L'Hôpital's Rule

THEOREM 6— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

37. $\lim_{x \to \infty} (\ln 2x - \ln (x + 1))$

39. $\lim_{x \to 0^+} \dfrac{(\ln x)^2}{\ln (\sin x)}$

41. $\lim_{x \to 1^+} \left(\dfrac{1}{x - 1} - \dfrac{1}{\ln x}\right)$

45. $\lim_{t \to \infty} \dfrac{e^t + t^2}{e^t - t}$

46. $\lim_{x \to \infty} x^2 e^{-x}$

47. $\lim_{x \to 0} \dfrac{x - \sin x}{x \tan x}$

48. $\lim_{x \to 0} \dfrac{(e^x - 1)^2}{x \sin x}$

Indeterminate Powers and Products

63. $\lim_{x \to 0^+} x^2 \ln x$

64. $\lim_{x \to 0^+} x (\ln x)^2$

65. $\lim_{x \to 0^+} x \tan\left(\dfrac{\pi}{2} - x\right)$

66. $\lim_{x \to 0^+} \sin x \cdot \ln x$

Indeterminite Powers

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

53. $\lim_{x \to \infty} (\ln x)^{1/x}$

54. $\lim_{x \to e^+} (\ln x)^{1/(x-e)}$

59. $\lim_{x \to 0^+} x^x$

60. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$

61. $\lim_{x \to \infty} \left(\frac{x+2}{x-1}\right)^x$

62. $\lim_{x \to \infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$

a. Use l'Hôpital's Rule to show that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$