Lecture

Sunday, November 22, 2020 2:56 PM

4.3 Monotonic Functions and the First Derivative Test

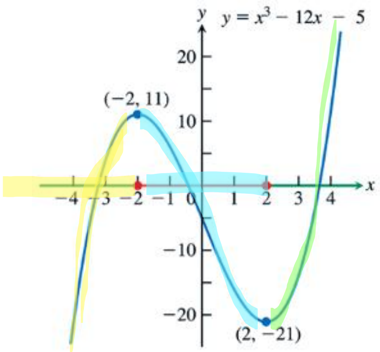
Increasing Functions and Decreasing Functions

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

$f' > 0 \Rightarrow f$ is increasing
$f' < 0 \Rightarrow f$ is decreasing

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.



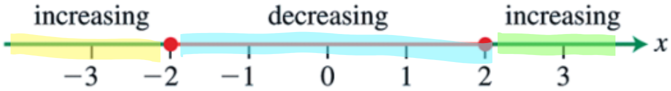
$f'(x) = 3x^2 - 12$
$= 3(x^2 - 4) = 0 <$ $\boxed{x = 2}$, $\boxed{x = -2}$ critical point.

(f' = undef $\rightsquigarrow$ no points)

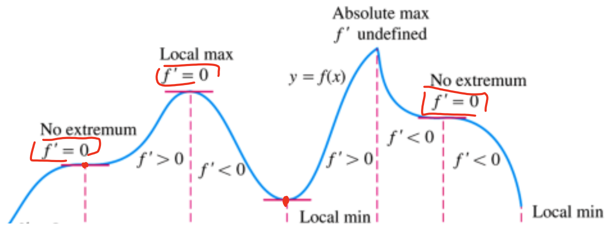
critical pts.

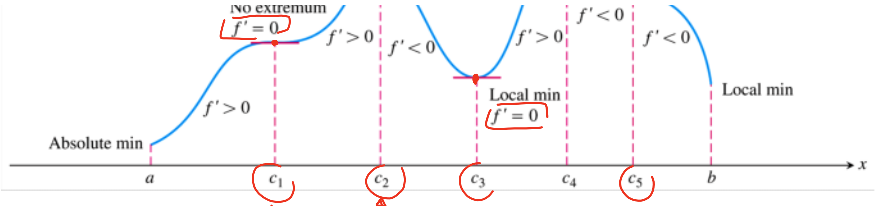
	$-\infty$		-2		2		$+\infty$
f'		$+$	0	$-$	0	$+$	
f		↗		↘		↗	

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing



The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.





no local extremum value.

local max

local min

no local ext. value

c_1: f' + ϕ + — f' does not change sign

c_2: f' + ϕ — — f' change(s) sign! at $x = c_2$

c_3: f' — ϕ + — f' changes sign.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

(1) f': − | c | +; f: ↘ ↗ — $x = c$ local min

(2) f': + | c | −; f: ↗ ↘ — $x = c$ local max

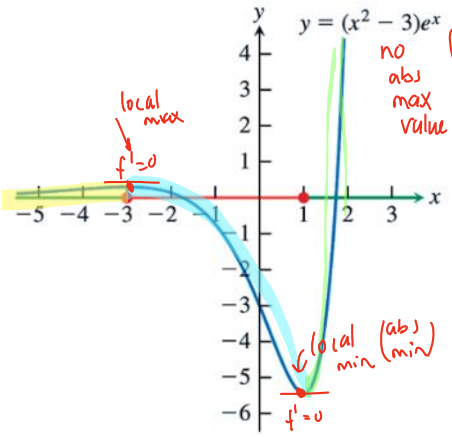
→ first derivative changes its sign.

(3) f': + | c | +, or − | c | − — first does not change sign. → no local extremum pts.

EXAMPLE 3 Find the critical points of

$$f(x) = (x^2 - 3)e^x.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.



$f'(x) = 2x \cdot e^x + (x^2 - 3) \cdot e^x$

$= e^x (x^2 + 2x - 3)$

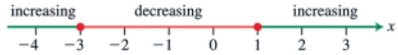
$f' = \underbrace{e^x}_{\neq 0} \cdot (x+3)(x-1) = 0$ → $x = -3$, $x = 1$ crit. points

(f' = undef → no such pt)

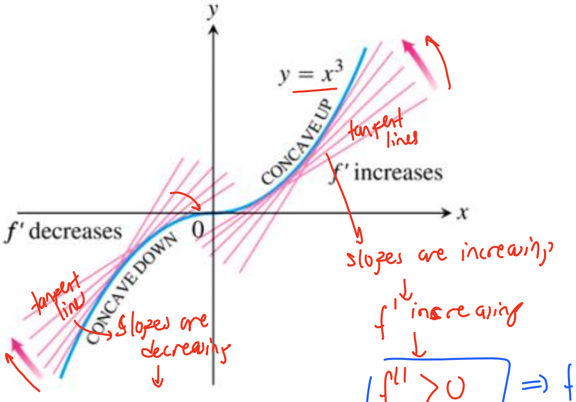
	$-\infty$		-3		1		$+\infty$
f'		+	0	−	0	+	
f		↗		↘		↗	

$x = -3$ local max; $x = 1$ local min

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing



4.4 Concavity and Curve Sketching



slopes are increasing
↓
f' increasing
↓
$\boxed{f''>0} \Rightarrow f$ conv up

slopes are decreasing
↓
f' = slope
↓
f' dec.
↓
$\boxed{f''<0} \Rightarrow f$ conv down

Summary

$f'>0$	$\rightarrow$	f inc
$f'<0$	$\rightarrow$	f dec

$f''>0$	$\Rightarrow$	f conv up
$f''<0$	$\Rightarrow$	f conv down

FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

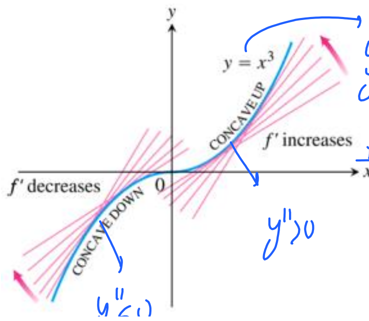
Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.



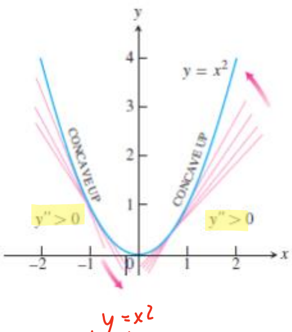
$y' = 3x^2$

$y'' = 6x = 0 \rightarrow x = 0$

		0	
f''	$-$		$+$
f	$\cap$		$\cup$

$y''>0$

$y''<0$



$y = x^2$

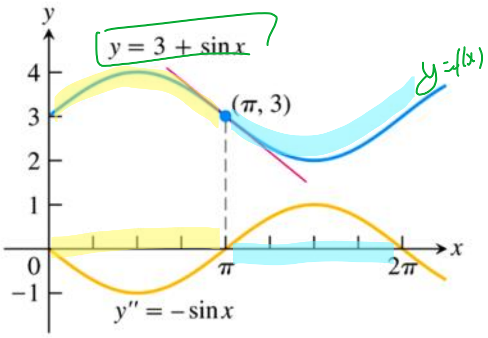
$y' = 2x$

$y'' < 0$

$y'' > 0$

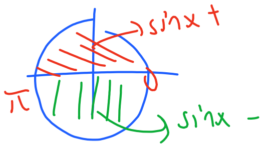
$y = x^2$
$y' = 2x$
$y'' = 2 \rightarrow$ always positive
$\downarrow$
f is always conv up

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.



y
$y' = \cos x$ on $[0, 2\pi]$
$y'' = -\sin x = 0$ → $x = 0$, $x = \pi$, $x = 2\pi$ on $[0, 2\pi]$

	0 – π	π – 2π
f''	−	+
f	∩	∪
	conv down	conv up.



DEFINITION A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

	c	
f''	−	+
f	∩	∪

OR

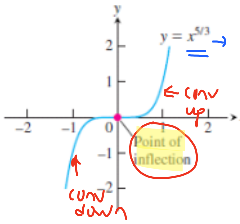
	c	
f''	+	−
f	∪	∩

conv. changes (f'' changes its sign) conv changes (f'' changes its sign)
$\downarrow$
$x = c$ → inflection point ← $x = c$

!

At a point of inflection $(c, f(c))$, either $\boxed{f''(c) = 0}$ or $\boxed{f''(c) \text{ fails to exist.}}$

$x = c$ inflection point → $f''(c) = 0$
OR
→ $f''(c) = $ undef



$y' = \frac{5}{3}x^{\frac{2}{3}}$

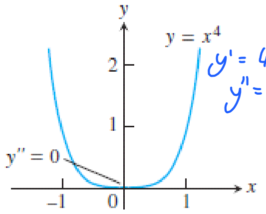
$y'' = \frac{10}{9}x^{-\frac{1}{3}} = \frac{10}{9}\frac{1}{\sqrt[3]{x}}$

y'' is undef. at $x=0$

y'': $-$ | 0 | $+$
y: $\cap$ undef $\cup$

$x=0$ conv. changes
$\Downarrow$
$\boxed{x=0}$ inflection point.

FIGURE 4.27 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).



$y' = 4x^3$

$y'' = 12x^2 = 0$
$x = 0$

y'': $+$ 0 $+$
$\cup$ $\cup$

y'' does not change sign
$\downarrow$
no inf. point.

FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

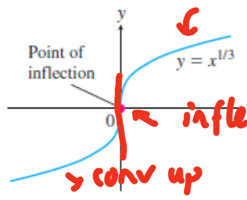
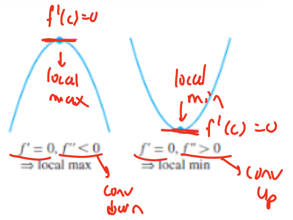


FIGURE 4.29 A point of inflection where y' and y'' fail to exist (Example 5).

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

conv down

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



Curve Sketching

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of in-

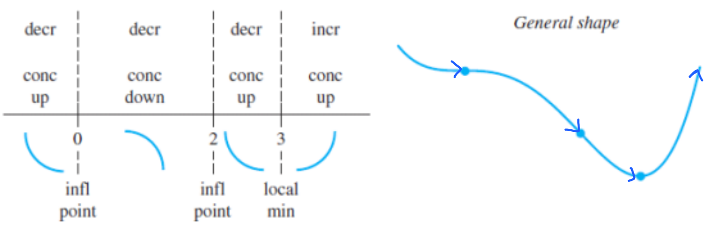
(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Exercise for you

The general shape of the curve is shown in the accompanying figure.



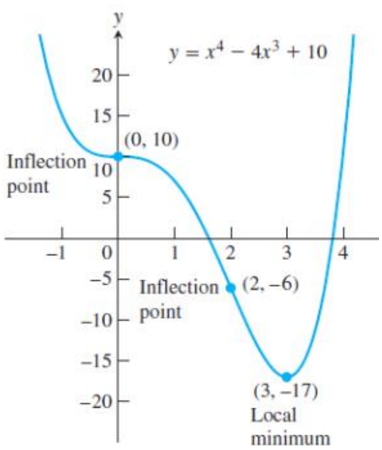


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist (see Section 2.6).
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

$\cancel{x^2 = -1}$

x-intercept: $y=0$, $\boxed{x=-1}$

y-intercept: $x=0$, $\boxed{y=1}$

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.* ($f' \to$ inc / dec $f'' \to$ conv up / conv down)

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

$$f'(x) = \frac{2(x+1)(1+x^2) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} = 0 \quad \boxed{x=1},\ \boxed{x=-1}$$

(f' = undef $\to$ no such point)

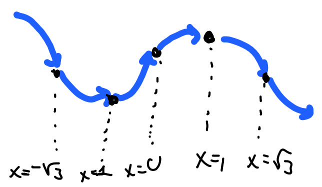
$$f''(x) = \frac{-4x\cdot(1+x^2)^2 - 2(1-x^2)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3} = 0 \quad \boxed{x=0},\ \boxed{x=-\sqrt{3}},\ \boxed{x=\sqrt{3}}$$

(f'' = undef $\to$ no such pnt)

3. *Behavior at critical points.*
4. *Increasing and decreasing.*

	$-\infty$		inf point $x=-\sqrt{3}$ ($-\sqrt{3}$)		local min $x=-1$ (-1)		inf pont $x=0$ (0)		local max $x=1$ (1)		inf point $x=\sqrt{3}$ ($\sqrt{3}$)		$+\infty$
f'		$-$		$-$	0	$+$		$+$	0	$-$		$-$	
f''		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
f		$\searrow$ $\cap$		$\searrow$ $\cup$		$\nearrow$ $\cup$		$\nearrow$ $\cap$		$\searrow$ $\cap$		$\searrow$ $\cup$	

5. *Inflection points.*



6. *Asymptotes.* : (x^2+2x+1)
   1. H.A : $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to -\infty} f(x) = 1$ → $\boxed{y=1}$ H.A
   2. V.A : no vertical asym
   3. ...

6. *Asymptotes.* :

② V.A : no vertical asym

③ O.A : no oblique asym.

$x \to \infty$ $\quad$ $x \to -\infty$

$\boxed{y=1}$ H.A

$f(x) = \dfrac{\overbrace{(1+x)^2}^{x^2+2x+1}}{1+x^2}$

Remark: $f(x) = \dfrac{P(x)}{Q(x)}$ ⇝ $\deg P(x) = \deg Q(x) + 1 \Rightarrow$ there is oblique asym.

$P(x) \mid Q(x)$, $B(x) = ax+b$, remainder $R(x)$; $y = ax+b$: $\boxed{y=ax+b}$ obl. asym.

7. The graph of f is sketched in Figure 4.31.

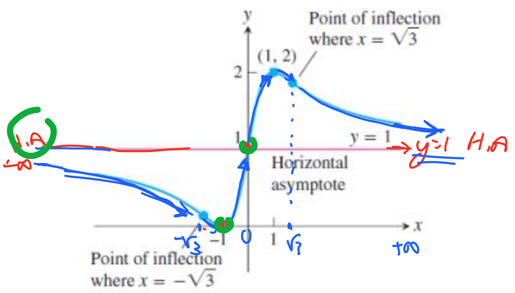


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2+4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2.
$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x}$$
$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}$$
$$f''(x) = \frac{4}{x^3}$$

3. *Behavior at critical points.*

4. *Increasing and decreasing.*

5. *Inflection points.*

6. *Asymptotes.*

7. The graph of f is sketched in Figure 4.32.

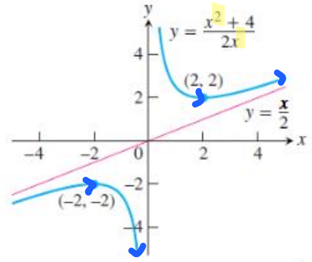
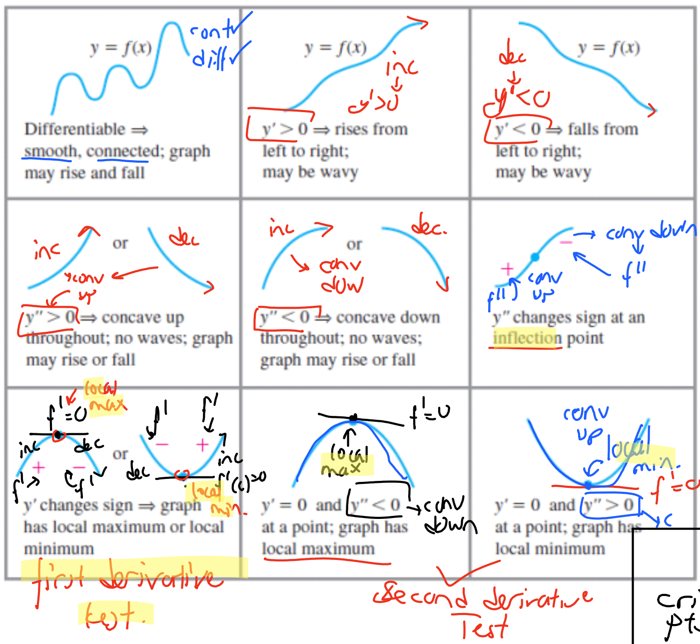


FIGURE 4.32 The graph of $y = \dfrac{x^2+4}{2x}$ (Example 9).

Graphical Behavior of Functions from Derivatives

Graphical Behavior of Functions from Derivatives



$f' > 0 \Rightarrow f$ inc $\quad f'' > 0 \Rightarrow f$ conv up
$f' < 0 \Rightarrow f$ dec $\quad f'' < 0 \Rightarrow f$ conv down

• $f''(c) = 0$ OR $f''(c) = $ undef $\quad$ $x = c$ is a candidate for an inflection pt
$\Downarrow$

	c	
f''	+	−
	∪	∩

	c	
f''	−	+
	∩	∪

conv down
conv up
conv. changes (f'' changes its sign)
$\Downarrow$
$\boxed{x = c}$ inflection pt.
conv up
conv down

• $f'(c) = 0$ OR $f'(c) = $ undef $\quad$ critical pts. $\quad$ $\boxed{x = c}$ is a candidate for local extremum pt

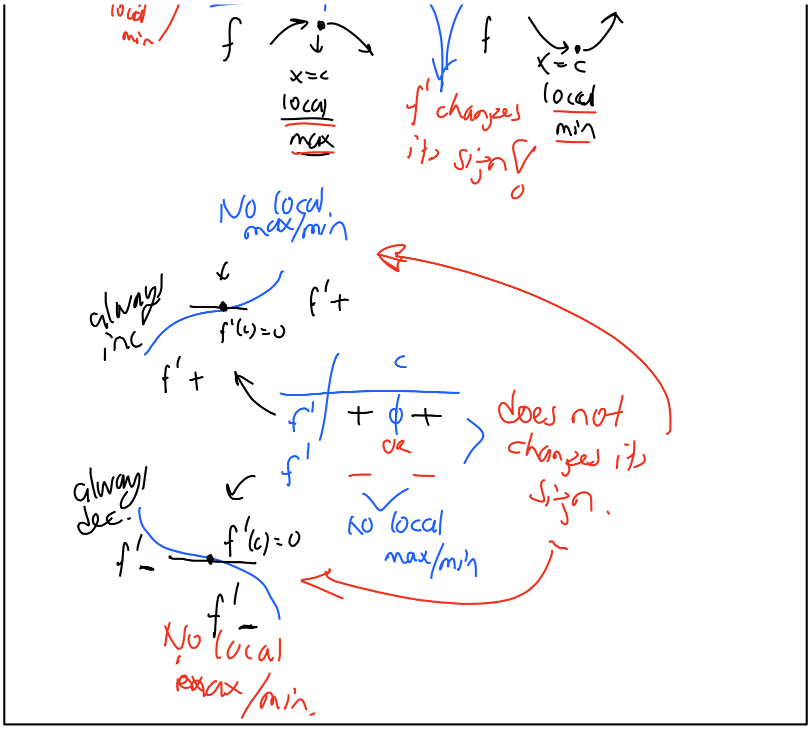
local max
local min

	c	
f'	+	−
f	↗	↘

$x = c$
local

	c	
f'	−	+
f	↘	↗

$x = c$
local



4.5 Indeterminate Forms and L'Hôpital's Rule

Determinate-Indeterminate Forms Table

Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty)\cdot(\infty) = \infty$
1^{∞}	
∞^0	
Use L'Hôpital's Rule	Do *Not* Use L'Hôpital's Rule

$\frac{0}{0} \neq 1$, $\frac{\pm\infty}{\pm\infty} \neq 1$, $\infty - \infty \neq 0$, $0\cdot\infty \neq 0$, $0\cdot\infty \neq \infty$, $0 \neq 0^0$, $1 \neq 1^{\infty}$, $1 \neq \infty^0$ — indeterminate power

$0^{-\infty} = \frac{1}{0^{\infty}} = \frac{1}{0} = \infty$

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\frac{-\infty}{-\infty} \text{ OR } \frac{\infty}{\infty} \text{ OR } \frac{0}{0} \rightsquigarrow \lim_{x \to a} \frac{f(x)}{g(x)} \overset{LH}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Warning: $\cancel{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$ ≠ $\frac{f'}{g'}$

Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

L'Hopital's Rule

- $\frac{0}{0} \Rightarrow$ directly apply L'H
- $\frac{\infty}{\infty} \Rightarrow$ directly apply L'H

$1^{\infty}, 0^0, \infty^0$

- $\frac{0}{0} \Rightarrow$ L'H directly apply
- $\frac{\infty}{\infty} \Rightarrow$ L'H directly apply
- $0 \cdot \infty \longrightarrow \frac{0}{\frac{1}{\infty}=0} = \frac{0}{0} \Rightarrow$ L'H

  $\hookrightarrow \frac{\infty}{\frac{1}{0}=\infty} = \frac{\infty}{\infty} \Rightarrow$ L'H
- $\infty - \infty \xrightarrow[\text{transfor}]{} \frac{0}{0}$ or $\frac{\infty}{\infty} \Rightarrow$ L'H

$\underbrace{1^{\infty},\ 0^{0},\ \infty^{0}}$

EXAMPLE 1 The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\lim_{x\to 0} \frac{3x - \sin x}{x} \sim \frac{0}{0}$ indeter. form

L'H $= \lim_{x\to 0} \frac{3 - \cos x}{1} = \lim_{x\to 0} 3 - \cancel{\cos x}^{\,1} = 2$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} \sim \frac{0}{0}$

L'H $= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \lim_{x\to 0} \frac{1}{2\sqrt{1+x}} = \frac{1}{2}$

(c) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$

(d) $\frac{0}{0} \sim \lim_{x\to 0} \frac{x - \sin x}{x^3} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \sim \frac{0}{0} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{+\sin x}{6x} \sim \frac{0}{0}$

$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cancel{\cos x}^{\,1}}{6} = \frac{1}{6}$

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}$.

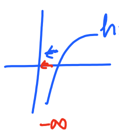
b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \sim \frac{\infty}{\infty}$

L'H $= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+} \sqrt{x} \ln x$

b) $\lim_{x\to 0^+} \sqrt{x} \ln x \sim 0 \cdot (-\infty)$



$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}} = x^{-1/2}} \sim \frac{-\infty}{\infty} \Rightarrow$ L'H

L'H $= \lim \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim -2x^{1/2} = 0$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2x^{1/2} = 0$$

a) $\lim_{x\to\infty} x.\sin\frac{1}{x} \sim \infty.0$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \sim \frac{0}{0} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos\frac{1}{x}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \cos 0 = 1$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right). \sim \underbrace{\infty - \infty}_{\text{indet. form.}} \neq 0$$

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} \sim \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{1-\cos x}{\sin x + x.\cos x} \sim \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{\sin x \to 0}{\cos x + \cos x + x(-\sin x)}$$

$$= \frac{0}{2} = 0$$

Indeterminite Powers $1^\infty, 0^0, \infty^0$

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

$$\lim_{x\to a} f(x)^{g(x)} \sim 1^\infty,\ 0^0,\ \infty^0$$

$$\lim_{x\to a} \ln f(x)^{g(x)} = \lim_{x\to a} g(x)\ln f(x) = \text{- - - - - -} = \boxed{A}$$

$(0.\infty) \to \frac{0}{0}, \frac{\infty}{\infty} \Rightarrow L'H$

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln f(x)^{g(x)}} = e^{\lim_{x\to a}\ln f(x)^{g(x)}} = \boxed{e^{A}}$$

$$= e^A$$

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} \sim 1^\infty$$

$$\lim_{x\to 0^+} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{x}.\ln(1+x) \sim \infty.0$$

$$= \lim \frac{\ln(1+x)}{x} \sim \frac{0}{0}$$

$$\lim_{x\to 0^+}$$

$$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \sim \frac{0}{0}$$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{(1+0)} = \boxed{1}$$ result of the question with ln

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^{\boxed{1}} = e$$

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right) \to 1^{\infty}$$ indeter. power.

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty} x \cdot \ln\left(1+\frac{1}{x}\right) \to \infty \cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \to \frac{0}{0}$$ Apply LH

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$$ ← resuınt of the question with ln

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x} = e^{\boxed{1}} = e$$