Problem Solving_sec03

8 Kasım 2020 Pazar 21:53

Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

28. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

38. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

39. $f(x) = \frac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

40. $g(x) = e^{-x^2}, \quad -2 \le x \le 1$

Recall) Find abs max/min of f(x) over [a,b] ?

Closed Interval Method:

① Critical Points: $f'(x) = 0$ or $f'(x) = \text{undef}$ → c_1, c_2, c_3 cri. pts.

② List:

cr. pts. $c_1 \to f(c_1) =$, $c_2 \to f(c_2) =$, $c_3 \to f(c_3) =$

end points $a \to f(a) =$, $b \to f(b) =$

choose the biggest as → abs max

" " smallest as → abs min

---

(28) $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

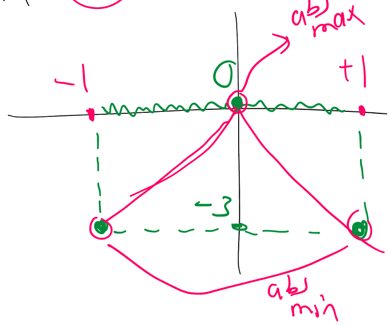
① Critical pts: $h'(x) = -3 \cdot \frac{2}{3} x^{-1/3} = -2 \frac{1}{\sqrt[3]{x}}$ → ~~$h' = 0$~~ → no pt. → $h' = \text{undef} \to \boxed{x = 0}$ only cr. pt.

② List

cr. pt. → $x = 0 \longrightarrow h(0) = -3 \cdot 0 = 0$ → abs max value

end pts. $x = -1 \longrightarrow h(-1) = -3 \cdot 1 = -3$

$x = +1 \longrightarrow h(1) = -3 \cdot 1 = -3$ → abs min value



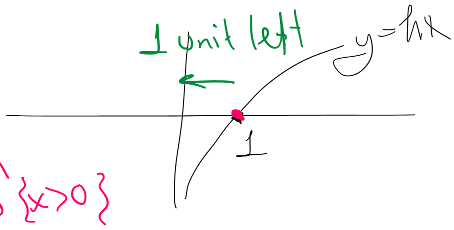
$h(x) = \ln(x+1), \quad 0 \le x \le 3$

① Cr. pts: $h'(x) = \frac{1}{x+1}$ → ~~$h' = 0$~~ → no pt! → $h' = \text{undef} \to$ ~~$x = -1$~~ → not included in the interval [0,3]

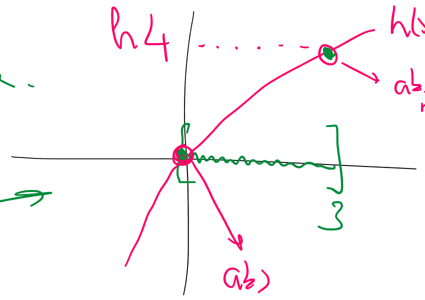
② List

$x=0 \longrightarrow h(0) = h(1) = 0$ ← abs min

$x=3 \longrightarrow h(3) = h(4)$ ← abs max.



Domain
ln x → {x>0}



---

(40) $g(x) = e^{-x^2}, \quad -2 \le x \le 1$

Recall
$(e^{f(x)})' = e^{f(x)} \cdot f'(x)$

① cr. pt: $g'(x) = e^{-x^2} \cdot (-2x)$

$= \dfrac{-2x}{e^{x^2}}$ → $g' = 0 \to \boxed{x=0}$

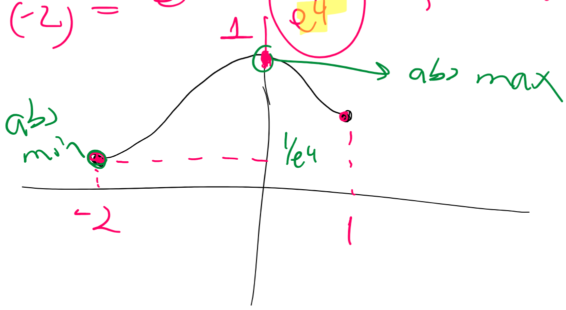
$e^{x^2} > 0$ → $g' =$ undef → no pt !

ok

② $x = 0 \longrightarrow g(0) = 1$ → biggest → abs max

$x = 1 \longrightarrow g(1) = e^{-1} = \frac{1}{e}$

$x = -2 \longrightarrow g(-2) = e^{-4} = \frac{1}{e^4}$ → smallest → abs min



In Exercises 45–56, determine critical points and domain endpoints for each function.

53. $y = \ln(x+1) - \tan^{-1} x$

Domain: $x+1>0$, $x>-1$, $(-1,\infty)$; $\mathbb{R}$

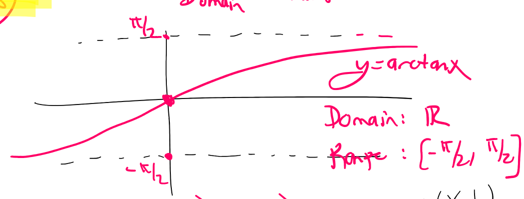
Domain of $y = (-1,\infty) \cap \mathbb{R}$
$= (-1,\infty)$

Recall

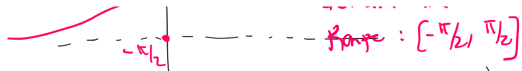
Domain of f = {all points at which f is defined}

Range of f = {all images of f(x)}

$f: \underset{\text{Domain}}{A} \to \underset{\text{Range}}{B}$



$x^2 + \cancel{x} - x - \cancel{x} \quad - \quad x(x-1)$ → $y' = 0 \to \boxed{x=0}, \boxed{x=1}$



$$y' = \frac{1}{x+1} - \frac{1}{x^2+1} = \frac{x^2+\cancel{1}-x-\cancel{1}}{(x+1)(x^2+1)} = \frac{x(x-1)}{(x+1)(x^2+1)}$$

$\to y'=0 \to \boxed{x=0}, \boxed{x=1}$

$\to y' = \text{undef} \to \boxed{\cancel{x=-1}} \to$ not included in the Domain.

List

$x=0 \longrightarrow g(0) = \overset{0}{\ln(1)} - \overset{0}{\arctan 0} = 0$

$x=1 \longrightarrow g(1) = \ln(2) - \underbrace{\arctan 1}_{\pi/4} = \ln 2 - \frac{\pi}{4}$

end $\left\{ \begin{array}{l} \lim\limits_{x\to -1} \longrightarrow \lim\limits_{x\to -1} g(x) = -\infty \\ \lim\limits_{x\to\infty} \longrightarrow \lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \cancel{\ln(x+1)} - \cancel{\arctan x} = \infty \end{array} \right.$

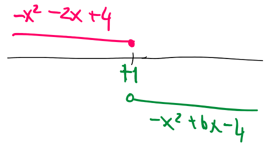
$\ln(x+1) \to \infty$, $\arctan x \to \pi/2$

no abs max
no abs min !

Local Extrema and Critical Points

In Exercises 57–64, find the critical points and domain endpoints for each function. Then find the value of the function at each of these points and identify extreme values (absolute and local).

63. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$



Domain: $(-\infty, \infty) = \mathbb{R}$

• Critical points:

$y' = \begin{cases} -2x-2 & x<1 \\ ? & x=1 \\ -2x+6 & x>1 \end{cases}$

$x=1 \longrightarrow y'(1) = $ def, exist ?

$\lim\limits_{x\to 1^+} \underbrace{y'(x)}_{-2x+6} \overset{?}{=} \lim\limits_{x\to 1^-} \underbrace{y'(x)}_{-2x-2}$

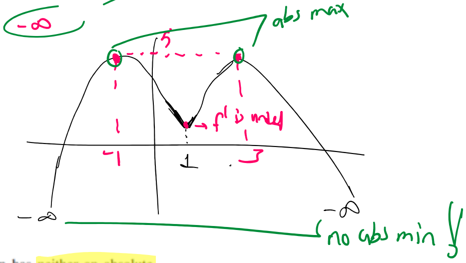
$4 \neq -4$
$\Downarrow$
y' is not defined at $\boxed{x=1}$
$\hookrightarrow \boxed{y'(1) = \text{undef}}$

$y'=0 \to -2x-2=0 \Rightarrow \boxed{x=-1} \to y'=0$
$\to -2x+6=0 \Rightarrow \boxed{x=3}$

List

cr. pt. $\left\{ \begin{array}{l} x=-1 \to y(-1) = 5 \\ x=3 \to y(3) = 5 \\ x=1 \to y(1) = 1 \end{array} \right.$ biggest $\to$ abs max value is 5

end $\left\{ \begin{array}{l} x\to-\infty \to \lim\limits_{x\to-\infty} y = -\infty \\ x\to+\infty \to \lim\limits_{x\to\infty} y = -\infty \end{array} \right.$ no abs min



In Exercises 67–70, show that the function has neither an absolute minimum nor an absolute maximum on its natural domain.

67. $y = x^{11} + x^3 + x - 5$ **68.** $y = 3x + \tan x$

69. $y = \dfrac{1 - e^x}{e^x + 1}$ **70.** $y = 2x - \sin 2x$

(67) $y = x^{11} + x^3 + x - 5 \to$ Domain $= (-\infty, \infty)$

69. $y = \frac{1 - e^x}{e^x + 1}$ **70.** $y = 2x - \sin 2x$

(67) $y = x^{11} + x^3 + x - 5 \to$ Domain $= (-\infty, \infty)$

Crit. pts: $y' = 11x^{10} + 3x^2 + 1 = 0 \to$ no pt; CR $=$ undef $\to$ no pt } no cr. points.

List
$x \to \infty \to \lim_{x\to\infty} x^{11}+x^3+x-5 = +\infty$
$x \to -\infty \to \lim_{x\to-\infty} x^{11}+x^3+x-5 = -\infty$
} no abs max, no abs min !

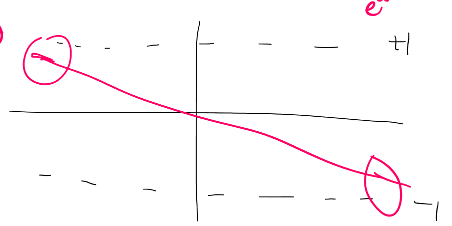
(69) $y = \frac{1-e^x}{1+e^x}$ Domain: $(-\infty, \infty)$

$$y' = \frac{-e^x(1+e^x) - (1-e^x)\cdot e^x}{(1+e^x)^2} = \frac{-2e^x}{(e^x+1)^2}$$
$\to \cancel{y'=0}$, $\cancel{y' = \text{undef}}$ $\to$ no cr. pts. !

$$\lim_{x\to\infty} \frac{1-e^x}{1+e^x} = \lim_{x\to\infty} \frac{\cancel{e^x}\left(\frac{1}{e^x} - 1\right)}{\cancel{e^x}\left(1 + \frac{1}{e^x}\right)} = -1$$

$$\lim_{x\to-\infty} \frac{1-e^x}{1+e^x} = 1$$

$e^{-\infty} = \frac{1}{e^{\infty}} = 0$

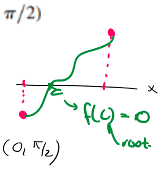


no abs max !
no abs min !

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

28. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

There is at least one root $\Rightarrow$ IVT



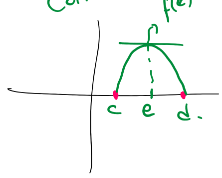
$r(\theta) = \tan\theta - \cot\theta - \theta$, $(0, \pi/2)$ → cont on $(0, \pi/2)$ ✓
$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$r\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} - \cot\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = -\frac{\pi}{4} < 0$
$r\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} - \cot\left(\frac{\pi}{3}\right) - \frac{\pi}{3} = \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{3} > 0$

} $\overset{\text{IVT}}{\Longrightarrow}$ there is at least one root $c \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ s.t. $f(c) = 0$

• To prove there is exactly one root
$\Downarrow$
Rolle's Thm: assumption:
✓ • $f(x)$ cont $[a,b]$
• $f(x)$ diff (a,b)
• $f(c) = f(d)$

Conclusion $f'(e) = 0$



$r(c) = 0$, Let's assume to the contrary that we have a second root $r(d) = 0$

$r(\theta)$ cont on $(0, \frac{\pi}{2})$ ✓
$r(\theta)$ diff on $(0, \frac{\pi}{2})$ ✓
$r(c) = r(d) = 0$ ✓
} Rolle's Thm $\Longrightarrow$ $r'(e) = 0$ ← this contradict to the fact that $r'(\theta) \geq 1$

$r(\theta) = \tan\theta - \cot\theta - \theta$
$r'(\theta) = \sec^2\theta + \csc^2\theta - 1 \geq 1 \Rightarrow \boxed{r'(\theta) \geq 1}$ Fact !

$\sec^2\theta = \frac{1}{\cos^2\theta} \geq 1$, $\csc^2\theta = \frac{1}{\sin^2\theta} \geq 1$
$0 \leq \cos^2\theta \leq 1$ $0 \leq \sin^2\theta \leq 1$

$\Downarrow$
another root can not happen
$\Downarrow$
$r(\theta)$ has exactly one root !

16. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Recall

assumptions
- f cont on $[a,b]$
- f diff on (a,b)

$\xrightarrow{\text{MVT}}$ $c \in (a,b)$ st $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

- f must be cont on $[0,2]$:

cont at $\boxed{x=0}$: $\lim\limits_{x \to 0^+} f(x) = f(0)$

$a = 3 \Rightarrow \boxed{a=3}$

cont at $\boxed{x=1}$: $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

$m+b = 5 \Rightarrow \boxed{m+b=5}$

- f must be diff on $(0,2)$

$\boxed{\text{at } x=1} \longrightarrow$ $f'(x) = \begin{cases} -2x+3 & 0 < x < 1 \\ m & 1 < x < 2 \\ & x = 1 \end{cases}$

diff ✓

$\lim\limits_{x \to 1^+} f'(x) \stackrel{\checkmark}{=} \lim\limits_{x \to 1^-} f'(x)$

$m = -2+3 = 1 \to \boxed{m=1} \to m+b=5$

$\boxed{b=4}$

30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

65. Show that $|\cos x - 1| \le |x|$ for all x-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

76. Use the same-derivative argument to prove the identities

a. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ **b.** $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

Recall:

1. $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$
2. $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$
3. $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$
4. $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$
5. $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$
6. $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$