Lecture

8 Kasım 2020 Pazar 20:53

4

4.1 Extreme Values of Functions on Closed Intervals

4.2 The Mean Value Theorem

Applications of Derivatives

4.1 Extreme Values of Functions on Closed Intervals

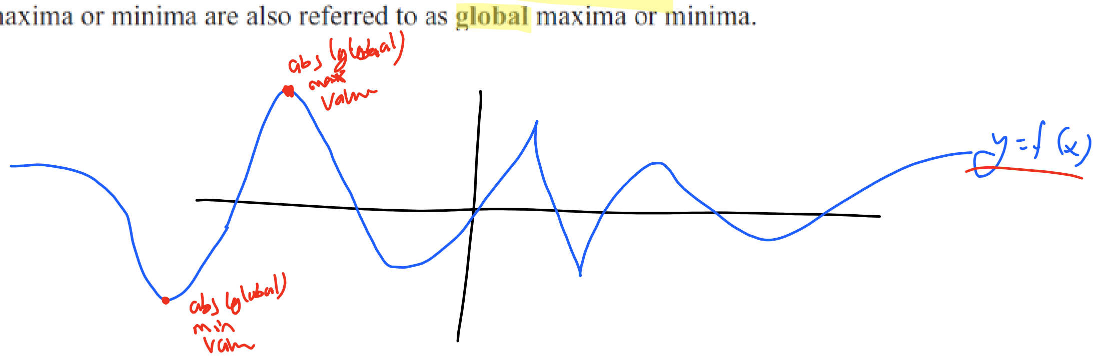
DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.



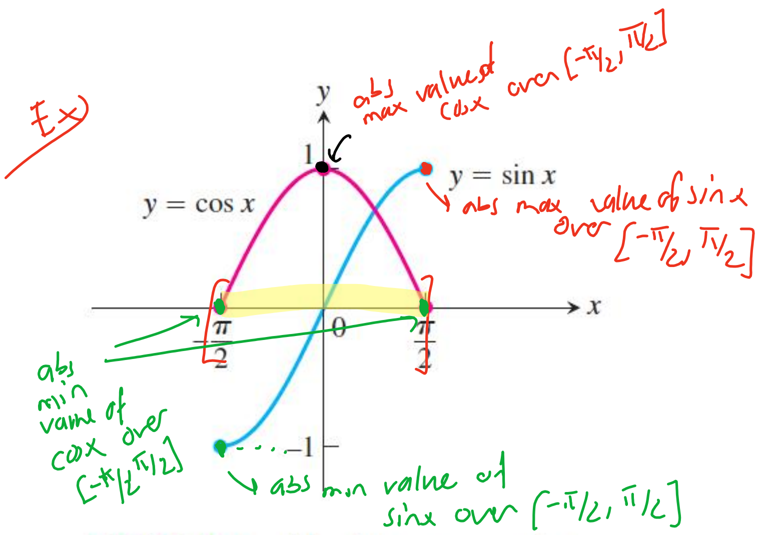


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend

the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Notice that a function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

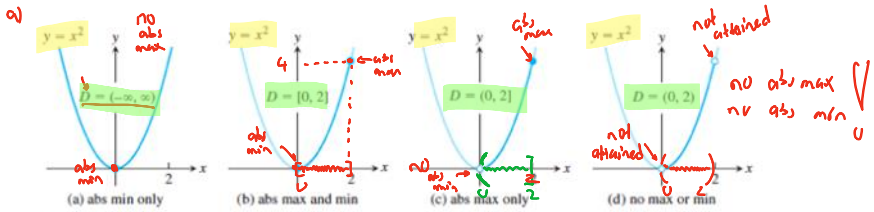


FIGURE 4.2 Graphs for Example 1.

assumption!

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

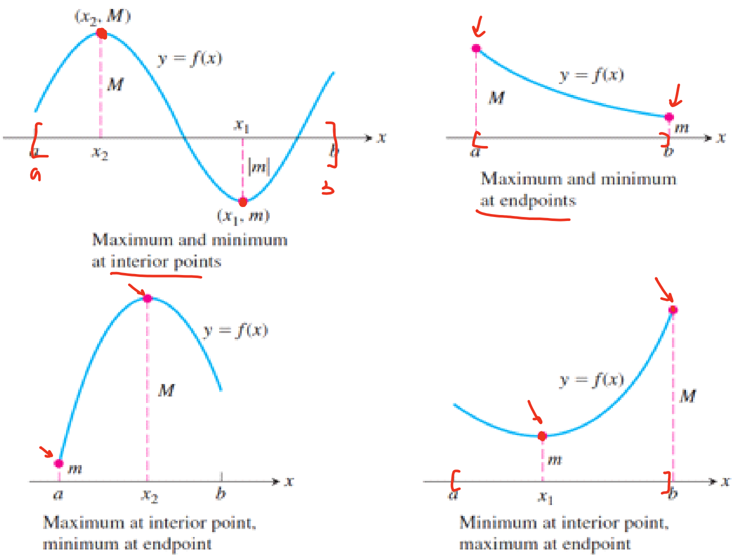


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

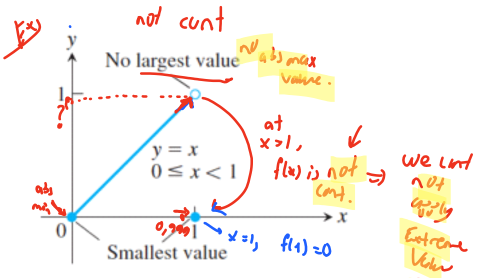


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

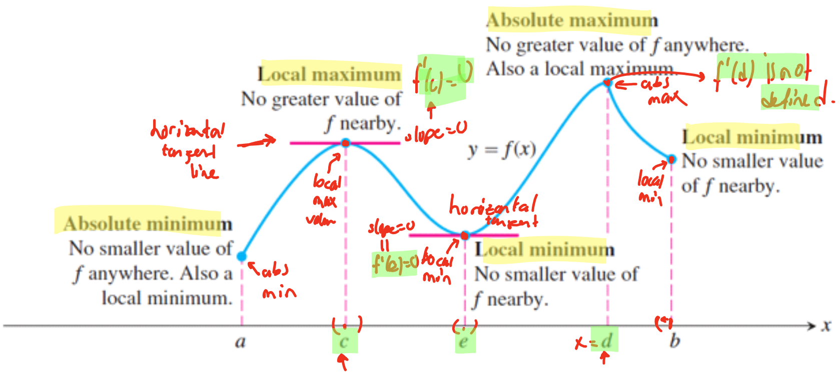


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

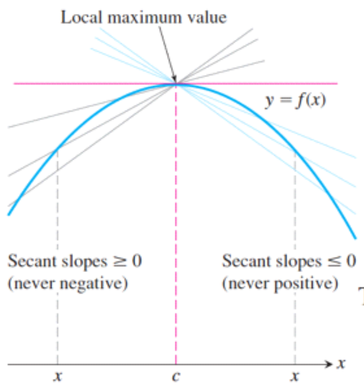


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \quad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \tag{1}$$

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \quad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \tag{2}$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

If f has local max/min at x=c + f' is defined } ⇒ $f'(c) = 0$ Thm.

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

local max/min
abs max/min

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

the points at which → $f' = 0$ or f' is undefined are called critical points

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

f is undefined

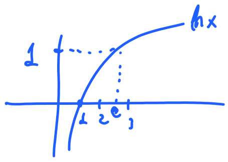
How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

abs max value abs min value

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

closed interval



$f(x) = 10x(2 - \ln x)$ is cont over $[1, e^2]$ → closed interval ✓

① critical points : $f'(x) = 10(2 - \ln x) + 10\cancel{x}(-\frac{1}{\cancel{x}})$

$= 20 - 10\ln x - 10$

$= 10 - 10\ln x$

$f'(x) = 10(1 - \ln x) = 0 \to \ln x = 1 \Rightarrow \boxed{x = e}$ only critical point.

f' undefined → no point !

② critical point: $x = e \to f(e) = 10e$ → biggest → abs max value

end point: $x = 1 \to f(1) = 20$

$x = e^2 \to f(e^2) = 0$ → smallest → abs min value

4.2 The Mean Value Theorem

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

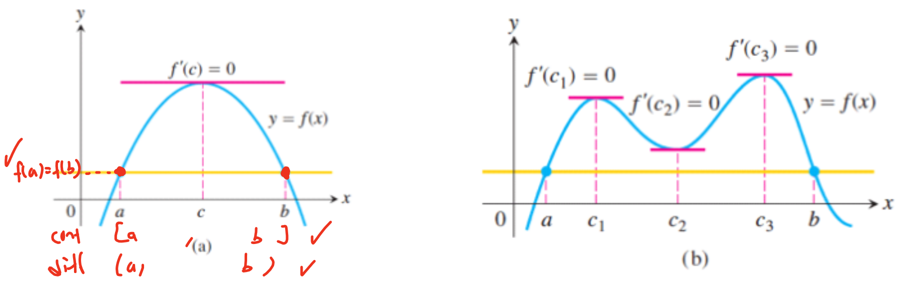
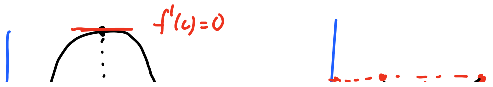
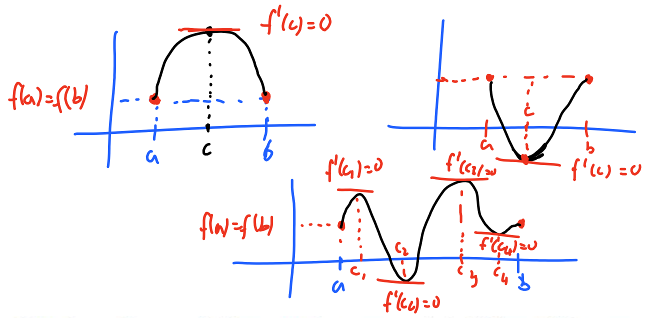


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

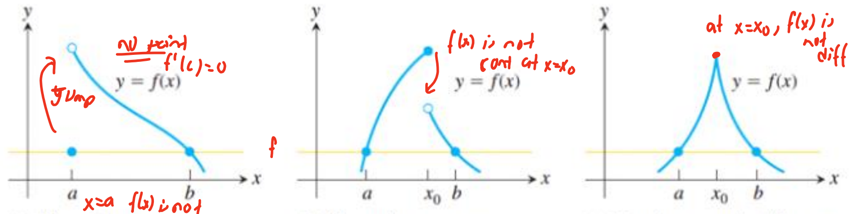
$f(x)$ cont $[a,b]$ $f(x)$ diff (a,b) $f(a) = f(b)$	⇒ Rolle's Thm	there is at least one $c \in (a,b)$ s.t. $f'(c) = 0$
assumption		conclusion.





The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

What if one of the assumptions does NOT hold? Rolle's Thm does Not apply



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

EXAMPLE 1 Show that the equation

$$\underbrace{x^3 + 3x + 1}_{f(x)} = 0$$

Rolle's Thm

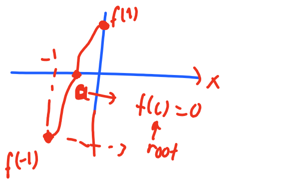
has exactly one real solution.

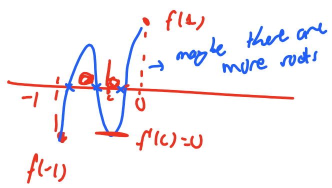
$$\boxed{\begin{array}{l}\text{To find at least one root of the given eqn (there is)} \Rightarrow \text{Use IVT} \\ \text{To prove that there is exactly one root} \Rightarrow \text{Use Rolle's Thm}\end{array}}$$

IVT

$f(x) = x^3+3x+1$ cont everywhere (we can apply IVT)

$f(0) = 1 > 0$
$f(-1) = -3 < 0$
$\xrightarrow{\text{IVT}}$ there is at least $a \in (-1,0)$ s.t $f(a)=0$ ← root





proving exactly one root: (Rolle's Thm)

Let's assume that there exists one more root: $\boxed{f(b)=0}$ assumed this

Then $\boxed{f(a)=f(b)=0}$ ✓

$f(x) = x^3+3x+1$ cont everywhere ✓, diff everywhere ✓ $\xrightarrow{\text{Rolle's Thm}}$ there is at least one $c \in (a,b)$ s.t $\boxed{f'(c)=0}$

ended up with a contradiction

$f'(x) = 3x^2+3 > 0$ always positive.

this contradicts the fact that f' is always positive

so there can not be

$f'(x) = 5x^4 + 3 > 0$
always positive.

⟵ f' is always positive
↓
so there can not be another root ~~$f(b)=0$~~
⇓
there should be only one root!

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\boxed{\frac{f(b) - f(a)}{b - a} = f'(c).} \qquad (1)$$

assumption

$$\boxed{\begin{array}{l} f(x) \text{ cont } [a,b] \\ f(x) \text{ diff } (a,b) \end{array}} \xRightarrow{\text{MVT}} \boxed{\begin{array}{l} \text{there is } c \in (a,b) \text{ s.t.} \\ f'(c) = \dfrac{f(b)-f(a)}{b-a} \end{array}}$$

conclusion

slope of tangent line = slope of secant line
⇓
tangent // secant line

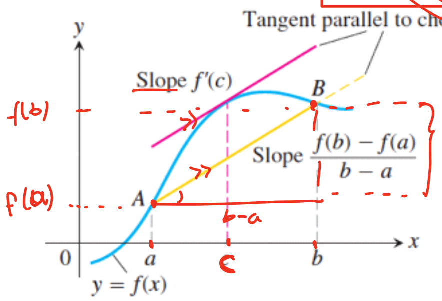


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to chord AB.

tangent line // secant line

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.18).

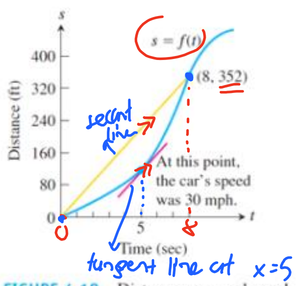


FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

avg velocity = $\dfrac{f(8)-f(0)}{8-0} = \dfrac{352-0}{8} = 44$ ft/sec.
|| rate of change of $f(t)$ over $[0,8]$
slope of secant line

$f(t)$ cont and diff
⇓ MVT
there is $c \in (0,8)$ s.t. $f'(c) \overset{?}{=} \dfrac{f(8)-f(0)}{8-0} = 44$
$c = 5$: slope of tangent line // slope secant line

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

$\boxed{f(x) = C \leftarrow \text{constant}}$

Quick proof: $f'(c) = \dfrac{f(b)-f(a)}{b-a}$
$f'(c) = 0$ for all x → $f(b)=f(a)$ for all values
⇓
$f(x) = $ constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

$f'(x) = g'(x) \Rightarrow \boxed{f(x) = g(x) + C}$

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

$f'(x) = \sin x \Rightarrow f(x) = -\cos x + C$ — to find this constant

$f(0) = 2$

$f(0) = -\underbrace{\cos 0}_{1} + c = 2$

$\boxed{C = 3}$

$f(x) = \cos x + 3$

Natural Exponential Function e^x	Natural Logarith Func. $\ln x$
(1) $e^a \cdot e^b = e^{a+b}$	(1) $\ln(a \cdot b) = \ln a + \ln b$
(2) $\frac{e^a}{e^b} = e^{a-b}$	(2) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
(3) $(e^a)^b = e^{a \cdot b}$	(3) $\ln x^n = n \cdot \ln x$
(4) $e^{-x} = \frac{1}{e^x}$	$\ln 1 = 0$

Recall

Exponential fnc. : a^x, $a \in \mathbb{R}$

logarithm fnc : $\log_a x$

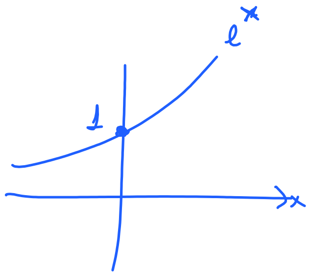
$\boxed{a = e}$

$a^x = e^x$

$\log_a x = \log_e x = \ln x$

Natural exp and logarithm.

$e^x \xrightarrow{\text{inverse func.}} \ln x$

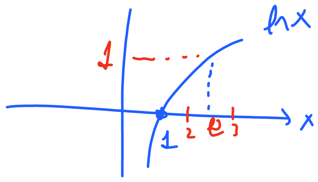


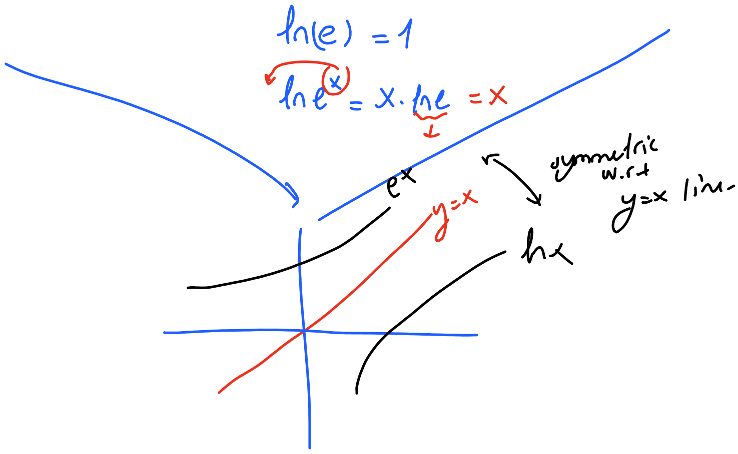
$\ln(e^x) = x$

$e^{\ln x} = x$

$\ln(e) = 1$

$\ln e^x = x \cdot \underbrace{\ln e}_{1} = x$





Laws of Exponents

The laws of exponents for the natural exponential e^x are consequences of the algebraic properties of $\ln x$. They follow from the inverse relationship between these functions.

Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$