Problem Solving_sec01

Sunday, November 01, 2020 10:27 PM

3.9 Inverse Trigonometric Functions

Ex) $y = \frac{1}{\arcsin x}$, $y' = ?$

$y = (\arcsin x)^{-1}$

$y' = -1 \cdot (\arcsin x)^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$

Recall

$(\arctan x)' = \frac{1}{1+x^2}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\text{arcsec}\, x)' = \frac{1}{x\sqrt{x^2-1}}$

Ex) $y = x \cdot \arctan\sqrt{x}$

$y' = 1 \cdot \arctan\sqrt{x} + x \cdot \underbrace{\frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}}_{(\arctan(\sqrt{x}))'}$

39. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1}x, \quad x > 1$

Recall

$(\arctan f(x))' = \frac{1}{1+(f(x))^2} \cdot f'(x)$

$y = \arctan\sqrt{x^2-1} + \text{arcsec}\, x$, $x > 1$

$y' = \frac{1}{1+(\sqrt{x^2-1})^2} \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x + \frac{1}{x\sqrt{x^2-1}}$

36. $y = \cos^{-1}(e^{-t}) = \arccos(e^{-t})$

Recall

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dt} = \frac{-1}{\sqrt{1-(e^{-t})^2}} \cdot e^{-t} \cdot (-1)$

31. $y = \cot^{-1}\sqrt{t}$

32. $y = \cot^{-1}\sqrt{t-1}$

33. $y = \ln(\tan^{-1}x)$

34. $y = \tan^{-1}(\ln x)$

35. $y = \tan^{-1}(e^{x^2})$

36. $y = \cot^{-1}\left(\frac{1}{x^2}\right) + \cos^{-1}2x$

37. $y = \sec^{-1}(e^{-x})$

38. $y = e^{\tan^{-1}\sqrt{x^2+1}}$

(33) $y' = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2}$

(36) $y = \text{arccot}\left(\frac{1}{x^2}\right) + \arccos 2x$

$y' = -\frac{1}{1+(\frac{1}{x^2})^2} \cdot (-2x^{-3}) \quad - \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$

(37) $y = \operatorname{arccot}\left(\frac{1}{x^2}\right) + \arccos(2x)$

$$y' = \frac{-1}{1+\left(\frac{1}{x^2}\right)^2} \cdot (-2x^{-3}) + \frac{-1}{\sqrt{1-(2x)^2}} \cdot 2$$

(38) $y = e^{\arctan(\sqrt{x^2+1})}$

$$y' = e^{\arctan(\sqrt{x^2+1})} \cdot \frac{1}{1+(\sqrt{x^2+1})^2} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

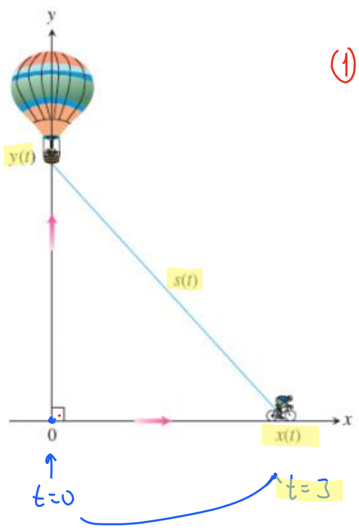
(39) $y = \operatorname{arcsec}(e^{-x})$

$$y' = \frac{1}{e^{x} \cdot \sqrt{(e^{-x})^2-1}} \cdot e^{-x} \cdot (-1).$$

3.10 Related Rates

application of derivative to real life.

33. A balloon and a bicycle A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



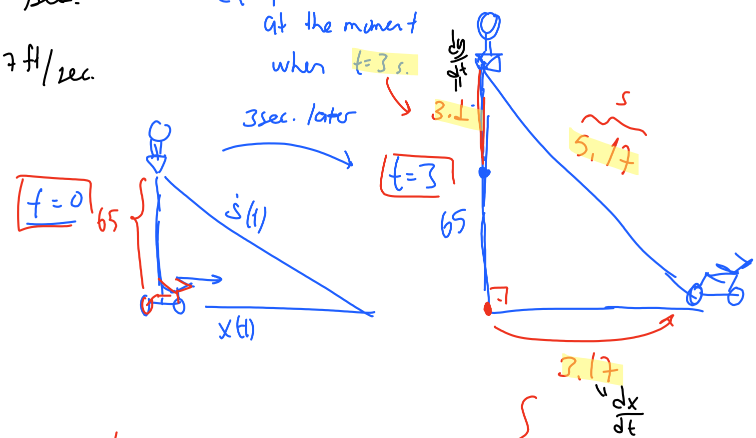
(1) Given that

$\frac{dy}{dt} = 1$ ft/sec.

$\frac{dx}{dt} = 17$ ft/sec.

Want to find

$\frac{ds}{dt}$ at the moment when $t=3$ s.



(2) Find a relating eqn:

$$x^2+y^2=s^2$$

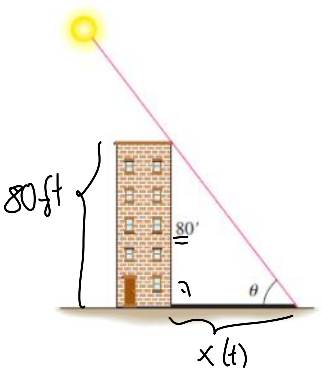
(3) $\frac{d}{dt}$: $\cancel{2}x \cdot \frac{dx}{dt} + \cancel{2}y \cdot \frac{dy}{dt} = \cancel{2}s \cdot \frac{ds}{dt}$

insert the particular moment $t=3$

$\frac{dx}{dt}=17$, $\frac{dy}{dt}=1$, $y=68$, $x=3.17$, $s=5,17$

$$3.17 \cdot 17 + 68 \cdot 1 = 5,17 \cdot \frac{ds}{dt} \Rightarrow \frac{ds}{dt} = 11 \text{ ft/sec.}$$

40. A building's shadow On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



(1) Given that

$$\left.\frac{d\theta}{dt}\right|_{x=60ft} = 0.27^{\circ}/\text{min} \quad \text{degrees}$$

Want to find

$$\left.\frac{dx}{dt}\right|_{x=60\text{ ft}} = ?$$

0.27 degrees/min ⇝ ? rad/min

360° — 2π

$0{,}27^{\circ}$ — a

$$a = \frac{2\pi \cdot 0{,}27}{360} = \boxed{\frac{\pi}{180} \cdot 0{,}27 \text{ rad/min}}$$

(2) relating eqn $\theta \leftrightarrow x$

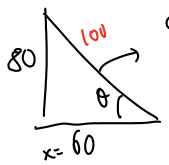
$$\tan\theta = \frac{80}{x}$$

(3) $\frac{d}{dt}\Big($

$$\sec^2\theta \cdot \frac{d\theta}{dt} = 80 \cdot -\frac{1}{x^2} \cdot \frac{dx}{dt}$$

insert the moment when $x=60ft$

at the moment when $x=60$

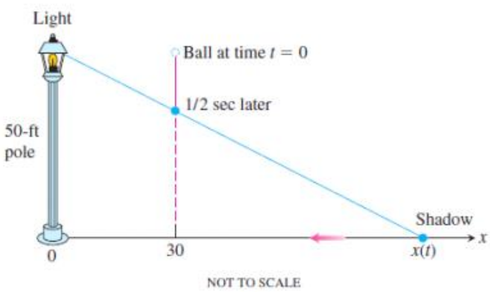


$\cos\theta = \frac{3}{5}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$

$$\left(\frac{5}{3}\right)^2 \cdot \frac{\pi}{180} \cdot 0{,}27 = 80 \cdot \frac{-1}{(60)^2} \cdot \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -2\pi \text{ ft/min}$$

x decreases in time at a rate of 2π ft/min.

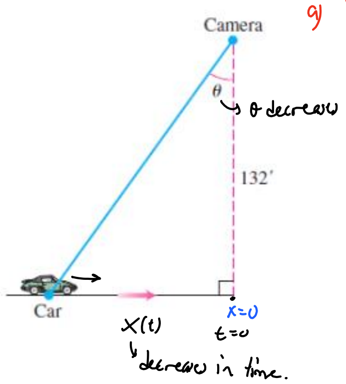


39. A moving shadow A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)

Exercise

38. Videotaping a moving car You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?

$x=0$



a) Given that

$\frac{dx}{dt} = -264$ ft/sec

x is decreasing

Want to find

$\left.\frac{d\theta}{dt}\right|_{x=0} = ?$

relating eqn:

$$\tan\theta = \frac{x}{132}$$

$\frac{d}{dt}$ ( ) $\frac{d}{dt}$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{132} \cdot \frac{dx}{dt}$$

insert the particular moment $x=0$, $\theta=0$, $\cos\theta$

$\theta=0 \to \cos 0 = 1$

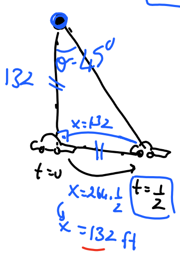
$\sec 0 = \frac{1}{\cos 0} = 1$

$$\underbrace{\sec^2 0}_{1} \cdot \frac{d\theta}{dt} = \frac{1}{132} \cdot (-264)$$

$$\frac{d\theta}{dt} = -2 \text{ rad/sec}$$

θ decrease in time.

b) moment when $\boxed{t=\frac{1}{2}}$ $\to$ $\frac{d\theta}{dt} = ?$



$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{132} \cdot \frac{dx}{dt}$$

$x=132$

$\theta = 45^\circ = \frac{\pi}{4}$

θ increases

$$\underbrace{\sec^2\frac{\pi}{4}}_{2} \cdot \frac{d\theta}{dt} = \frac{1}{132} \cdot (-264) \Rightarrow \frac{d\theta}{dt} = \mp 1 \text{ rad/sec.}$$

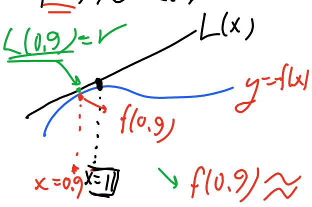
$$\sec^2 45 = \frac{1}{\cos^2(45)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

3.11 Linearization and Differentials

Recall Linearization of $f(x)$ at $x=a$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) \approx L(x)$$



$f(0.9) \approx L(0.9)$

Ex: Find approximately $(0,9)^{0,9}$

$f(x) = x^x \rightsquigarrow f(0.9) = (0.9)^{0.9}$

Linearization of $f(x)$ at $x=1$

$$L(x) = f(1) + f'(1)(x-1)$$

$$L(x) = 1 + 1\,(x-1)$$

$$\boxed{L(x) = x}$$

$$f(x) \approx L(x)$$

$y = x^x$

$\ln y = x \cdot \ln x$

$\frac{d}{dx}$ $\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$\Rightarrow y' = x^x(\ln x + 1)$

$y'(1) = 1^1 \cdot (\ln 1 + 1) = 1 = f'(1)$

$y'(1) = 1'.(\underbrace{\ln 1}_{0}+1) = 1 = f'(1)$

$f(x) \approx L(x)$

$\underbrace{(0.9)^{0.9} = f(0.9) \approx L(0.9) = 0.9}$

$(0.9)^{(0.9)} \approx 0.9$

image of original fnc $f(x)$

image of $L(x)$

Ex: find approximately $\sqrt[3]{8,2}$

$f(x) = \sqrt[3]{x}$

Linearization at $x=8$

Linearization for Approximation

In Exercises 7–14, find a linearization at a suitably chosen integer near x_0 at which the given function and its derivative are easy to evaluate.

11. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

12. $f(x) = \dfrac{x}{x+1}, \quad x_0 = 1.3$

13. $f(x) = e^{-x}, \quad x_0 = -0.1$

closest integer

Linearization of $f(x)$ at $x=0$

$L(x) = \underbrace{f(0)}_{=} + \underbrace{f'(0)}_{=}(x-0)$

$L(x) = e^{-0} + (-1)\ (x-0)$

$f'(x) = e^{-x}.(-1)$

$f'(0) = -1$

$L(x) = 1 - x$

$f(-0.1) = e^{0.1} \approx L(0.1) = 1-(-0.1) = 1.1$

$e^{0.1} \approx 1.1$

15. Show that the linearization of $f(x) = (1+x)^k$ at $x=0$ is $L(x) = 1 + kx$.

$f'(x) = k(1+x)^{k-1}$

$f'(0) = k.(1)^{k-1} = k$

$f(0) = 1$

linearization of $f(x)$ at $x=0$

$L(x) = f(0) + f'(0)(x-0)$

$L(x) = 1 + k.x$

$\boxed{L(x) = 1 + k \cdot x}$

$f(0) = 1$

17. **Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

$f(x) \approx L(x)$

$(1+x)^k \approx 1 + kx$

a) $(1.002)^{50} = (1 + 0.002)^{50} \approx 1 + 50.(0.002) = 1.01$

approximately.

Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$\boxed{dy = f'(x)\, dx.}$$

$\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\, dx$

differential.

Derivatives in Differential Form

In Exercises 19–38, find dy.

34. $y = \ln\left(\frac{x+1}{\sqrt{x-1}}\right)$ $f(x)$

$y = \ln(x+1) - \ln(\sqrt{x-1})$

$dy = \underbrace{\left(\frac{1}{x+1} - \frac{1}{\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x-1}}\right)}_{f'(x)} . dx$

31. $y = e^{\sqrt{x}}$

$dy = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot dx$

38. $y = e^{\tan^{-1}\sqrt{x^2+1}}$

$dy =$

32. $y = xe^{-x}$

$dy =$

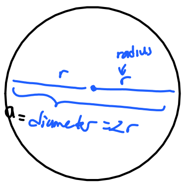
Estimating with Differentials

$y = f(x)$

$\underbrace{dy}_{\text{very small change in } y} = \underline{f'(x)}\, \underline{dx}$

52. The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-section area?

→ very small change in C.



$\underbrace{d}_{\text{diameter}} = 2r = 10 \text{ in.} \Rightarrow r = 5 \text{ in}$

$C(r) = 2\pi r$

$2 = dC = 2\pi \cdot dr \Rightarrow 2\pi dr = 2$

$\boxed{dr = \frac{1}{\pi} \text{ in.}}$ very small change in r.

- $d = 2r$

  $\boxed{dd} = 2 \cdot dr = 2 \cdot \frac{1}{\pi}$ in. (small change in diameter)

- cross section area $= A(r) = \pi r^2$

  small change in cross section area → $dA = 2\pi r \cdot dr$

  $= 2\pi \cdot 5 \cdot \frac{1}{\pi} = 10 \text{ in}^2$

$dr = \frac{1}{\pi}$ in $\rightsquigarrow$ $\underline{dA = 10 \text{ in}^2}$

55. **Tolerance** The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

  a. circumference? **b.** area?