Lecture_Template

Sunday, November 01, 2020 8:08 PM

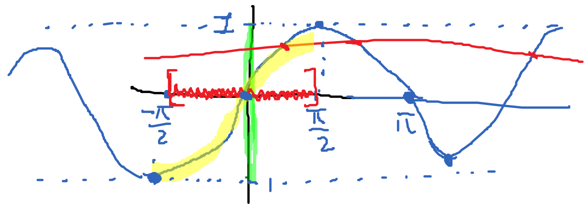
3.9 Inverse Trigonometric Functions

Recall

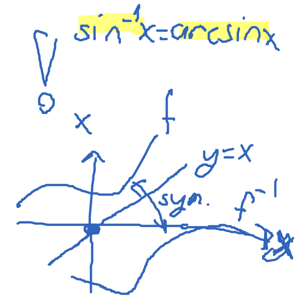
f is 1-to-1 then f^{-1} exists

$f: A \to B$ — domain, range

$f^{-1}: B \to A$



$\sin^{-1}x = \arcsin x$



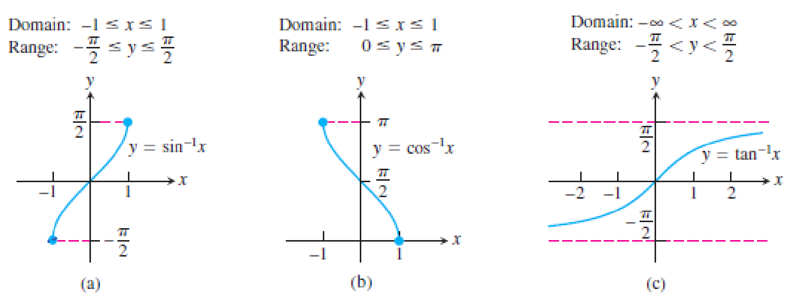
$\sin x : (-\infty, \infty) \to [-1, 1]$ — to obtain 1-to-1 fnc ⇒ restrict the domain

restrict

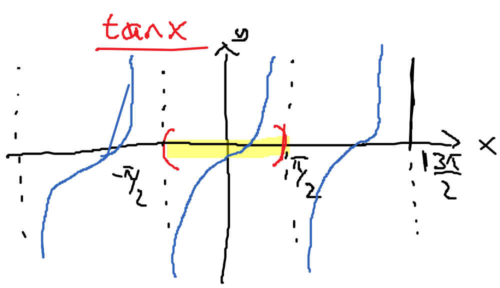
$\sin x : \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{1 \text{ to } 1 \checkmark} \to [-1, 1]$

$\arcsin x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain — Range



tanx



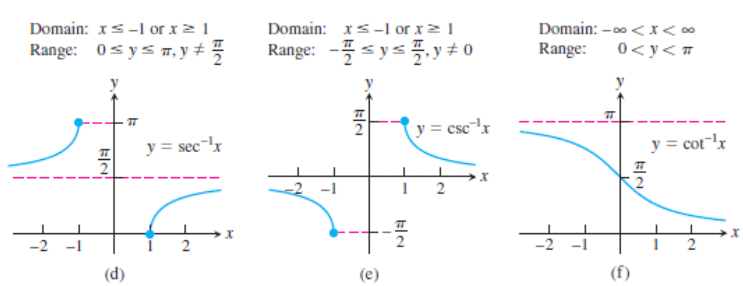
$\tan x : (\quad, \infty) \to (-\infty, \infty)$ — dmn, Range

restrict to have 1 to 1

$\tan x : \underbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}_{1 \text{ to } 1} \to (-\infty, \infty)$

$\arctan x : (-\infty, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain — Range



Definitions (3 of 4)

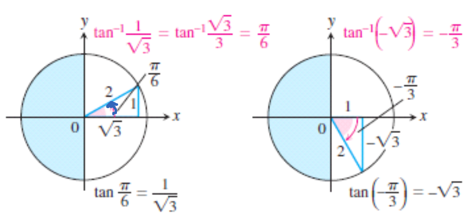
$y = \tan^{-1} x$ is the number in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1} x$ is the number in $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$ for which $\sec y = x$.

$y = \csc^{1} x$ is the number in $\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$ for which $\csc y = x$.

EXAMPLE 1 The accompanying figures show two values of $\tan^{-1} x$.



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

4. a. $\sin^{-1}\left(\dfrac{1}{2}\right)$ **b.** $\sin^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$

5. a. $\cos^{-1}\left(\dfrac{1}{2}\right)$ **b.** $\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)$

9. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

10. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

The Derivative of $y = \sin^{-1} u$

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}
$$

Example

1 $y = \sqrt{\arcsin(\ln 3x)}$

The Derivative of $y = \tan^{-1} u$

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
(\arctan x)' &= \frac{1}{1 + x^2}. && \tan(\tan^{-1} x) = x
\end{aligned}
$$

ule

Example

1) $y = \dfrac{1}{\arctan(e^{-x})} \Rightarrow$

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

Ex: Find y' for $y = \ln(\sec^{-1} x + \tan^{-1} x)$

Ex: Find y' for $y = \tan^{-1}(x - \sqrt{1-x^2})$

Ex: Find y' for $y = \sin^{-1}(\cos^{-1} x)$

36. $y = \cos^{-1}(e^{-t})$

3.10 Related Rates

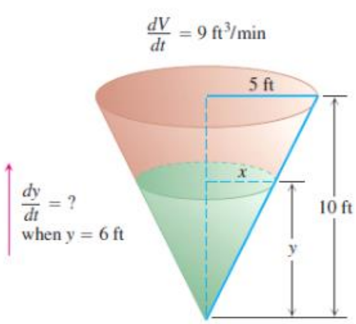


FIGURE 3.43 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.43 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

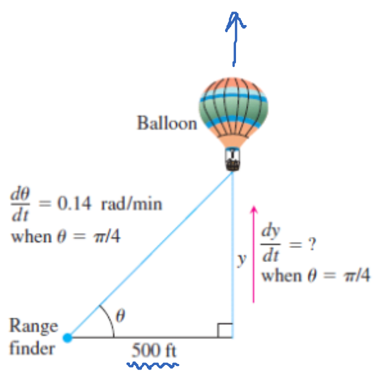


FIGURE 3.44 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

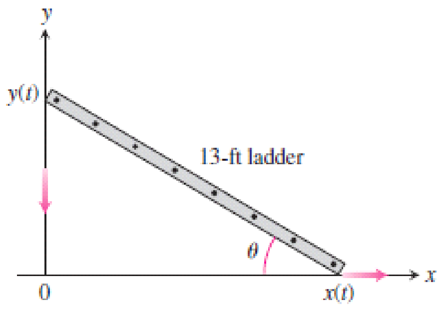
EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.44). The variables in the picture are

   $\theta =$ the angle in radians the range finder makes with the ground.

   $y =$ the height in feet of the balloon.



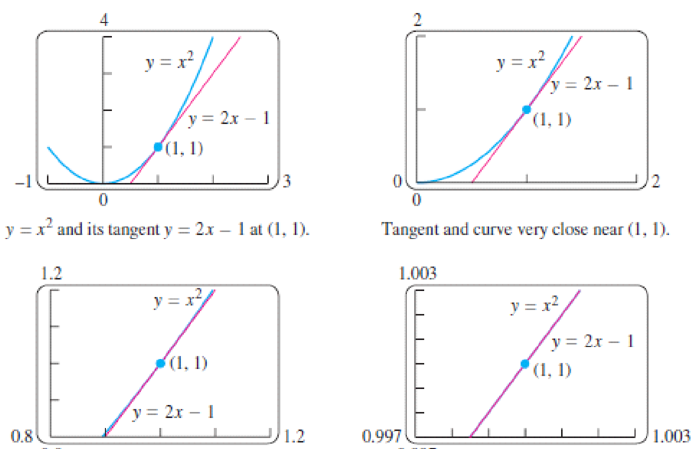
23. A sliding ladder A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

a. How fast is the top of the ladder sliding down the wall then?

b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

c. At what rate is the angle θ between the ladder and the ground changing then?

3.11 Linearization and Differentials



$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).

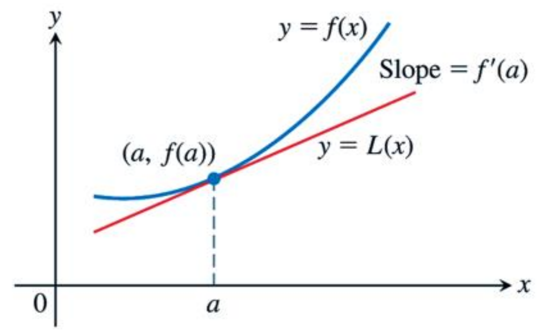
Tangent and curve very close throughout entire x-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

FIGURE 3.49 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

Figure 3.52

The tangent to the curve $y = f(x)$ at $x = a$
is the line $L(x) = f(a) + f'(a)(x - a)$.



DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.51).

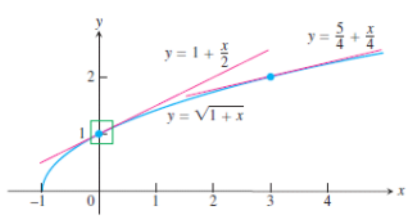


FIGURE 3.51 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.52 shows a magnified view of the small window about 1 on the y-axis.

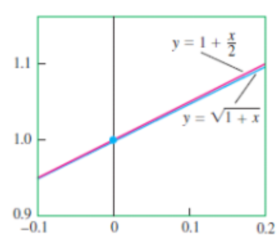


FIGURE 3.52 Magnified view of the window in Figure 3.51.

$\sqrt{1+x} \approx 1 + (x/2)$	
Approximation	**True value**
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.53).

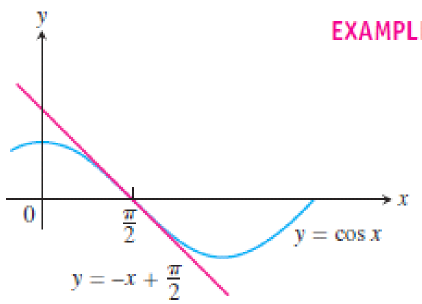


FIGURE 3.53 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

Ex: Find approximately $(0,9)^{0,9}$

Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

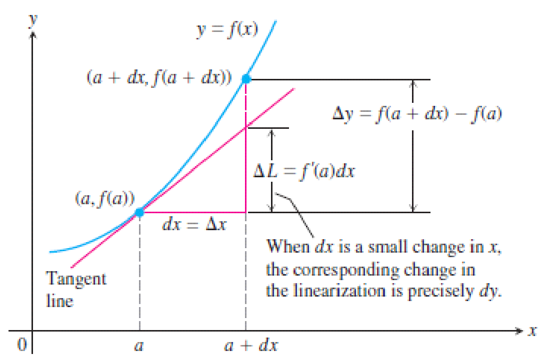


FIGURE 3.54 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 3.54 that Δy is approximately equal to the differential dy. Since

$$f(a + dx) = f(a) + \Delta y, \qquad \Delta x = dx$$

$$f(a + dx) \approx f(a) + dy$$

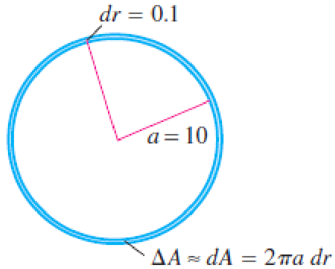


FIGURE 3.55 When dr is small compared with a, the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.55). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\, dr = 2\pi a\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately 102π m^2.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is 0.01π m^2, which is the difference $\Delta A - dA$.