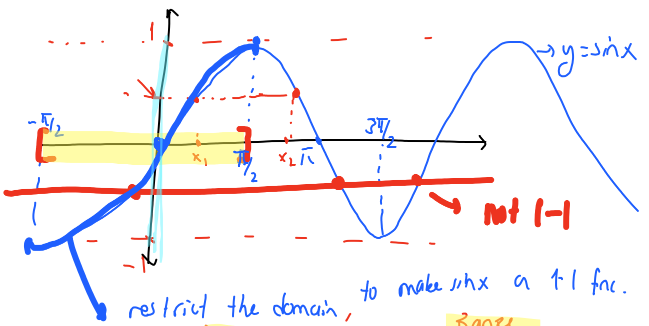
Lecture

Sunday, November 01, 2020 8:08 PM

3.9 Inverse Trigonometric Functions



Recall

Domain Range

$f: A \longrightarrow B$

$f^{-1}: B \quad A$

! to have an inverse fnc f has to be 1-1 and onto

restrict the domain, to make sinx a 1-1 fnc.

Domain Range

$\sin x : \underbrace{(-\infty, \infty)}_{\text{not 1-1}} \longrightarrow [-1, 1]$

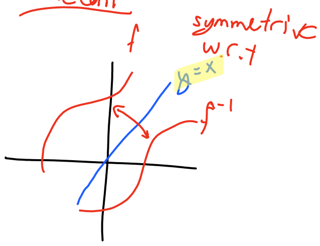
restrict the domain to have a 1-1 fnc.

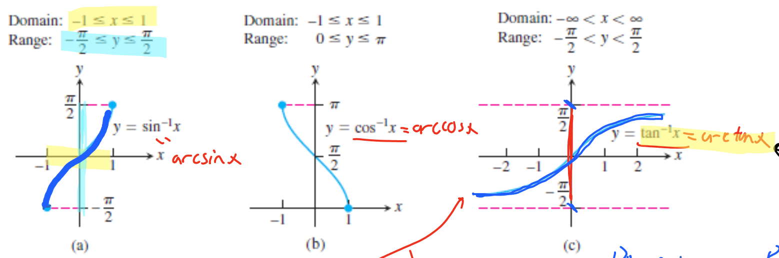
inverse sinx : $\underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{\text{it is 1-1 \& onto}} \longrightarrow [-1, 1]$ ⇒ it has an inverse fnc.

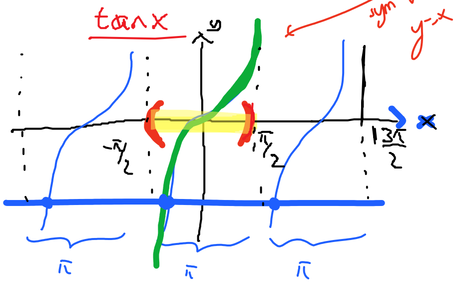
Domain Range-

$\arcsin x = \sin^{-1} x : [-1, 1] \longrightarrow \left[-\pi/2, \pi/2\right]$

Recall







Domain Range

tanx : $(-\infty, \infty)$ $(-\infty, \infty)$

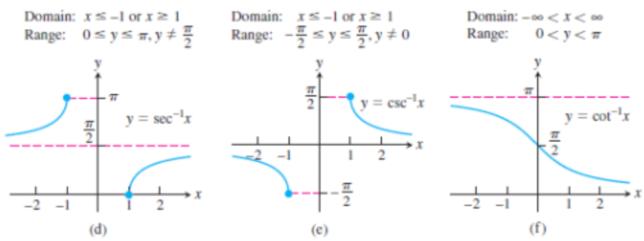
restrict to have a 1-1 fnc.

$\tan x : \underbrace{(-\pi/2, \pi/2)}_{\text{1 to 1 and onto}} \longrightarrow (-\infty, \infty)$

inverse

$\arctan x : (-\infty, \infty) \longrightarrow (-\pi/2, \pi/2)$

Domain Range.



Recall

$\tan^{-1} x \neq \frac{1}{\tan x}$

inverse fnc of $\tan x$ $= \arctan x$

Definitions (3 of 4)

range of arctan x

$\arctan x =$ $y = \tan^{-1} x$ is the number in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\boxed{\tan y = x.}$

$y = \arctan x$

$\tan y = \tan(\arctan x)$

inverse

$\boxed{\tan y = x}$

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1} x$ is the number in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ for which $\sec y = x$.

$y = \csc^{-1} x$ is the number in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ for which $\csc y = x$.

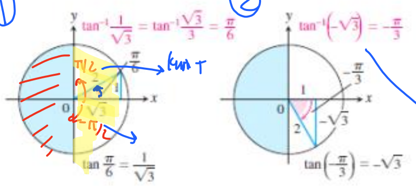
EXAMPLE 1 The accompanying figures show two values of $\tan^{-1} x$.

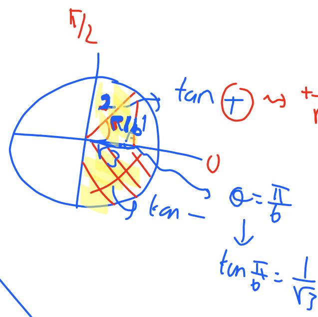
$\arctan\left(\frac{1}{\sqrt{3}}\right) = ?$

$\arctan\left(\frac{1}{\sqrt{3}}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$

$\tan\theta = +\frac{1}{\sqrt{3}} \Rightarrow \theta = \pi/6$



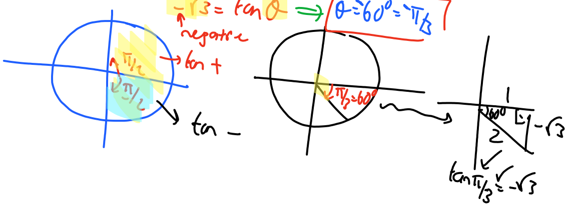


The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

$\arctan(-\sqrt{3}) = ?$ range of arctan

$\arctan(-\sqrt{3}) = \theta \in \left(-\pi/2, \pi/2\right)$

$-\sqrt{3} = \tan\theta \Rightarrow \boxed{\theta = 60^\circ = \pi/3}$ negative



4. a. $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

5. a. $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

9. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ **10.** $\sec\left(\cos^{-1}\frac{1}{2}\right)$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **12.** $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

The Derivative of $y = \sin^{-1} u$

Derivative of inverse fnc $= (f^{-1})'(b) = \frac{1}{f'(a)}$

$f(a) = b \Rightarrow a = f^{-1}(b)$

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{Theorem 3}$$

$$= \frac{1}{\cos(\sin^{-1} x)} \qquad f'(u) = \cos u$$

$$= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} \qquad \cos u = \sqrt{1 - \sin^2 u}$$

$$= \frac{1}{\sqrt{1 - x^2}}. \qquad \sin(\sin^{-1} x) = x$$

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f^{-1}(x) = \arcsin x$

$(\arcsin x)' = ?$

$(f^{-1})'_{(x)} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$

inverse

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos x = \sqrt{1 - \sin^2 x}$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

Example

$y = \sqrt{\arcsin(\ln 3x)}$

$$y' = \frac{1}{2\sqrt{\arcsin(\ln 3x)}} \cdot \frac{1}{\sqrt{1-(\ln 3x)^2}} \cdot \frac{1}{3x} \cdot 3$$

Recall

$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$$\arcsin(f(x))' = \frac{1}{\sqrt{1-f(x)^2}} \cdot f'(x)$$

The Derivative of $y = \tan^{-1} u$

$f(x) = \tan x$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{Theorem 3}$$

$$= \frac{1}{\sec^2(\tan^{-1} x)} \qquad f'(u) = \sec^2 u$$

$$= \frac{1}{1 + \tan^2(\tan^{-1} x)} \qquad \sec^2 u = 1 + \tan^2 u$$

$$\boxed{(\arctan x)' = \frac{1}{1 + x^2}} \qquad \tan(\tan^{-1} x) = x$$

inverse

Rule.

$f(x) = \tan x \rightsquigarrow f'(x) = \sec^2 x$

$f^{-1}(x) = \arctan x$

Recall

$\boxed{\sec^2 x = \tan^2 x + 1}$

Example

1) $y = \frac{1}{\arctan(e^{-x})} \Rightarrow y = (\arctan(e^{-x}))^{-1}$

$$y' = -(\arctan(e^{-x}))^{-2} \cdot \frac{1}{1+(e^{-x})^2} \cdot e^{-x} \cdot (-1)$$

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

$u(x) \rightsquigarrow f(x)$

$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \dfrac{1}{1+x^2}$

$(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$

$(\operatorname{arcsec} x)' = \dfrac{1}{x\sqrt{x^2-1}}$

$(\operatorname{arccosec} x)' = -\dfrac{1}{x\sqrt{x^2+1}}$

Ex: Find y' for $y = \ln(\sec^{-1}x + \tan^{-1}x)$

$y = \ln(\operatorname{arcsec} x + \arctan x)$

$y' = \dfrac{1}{\operatorname{arcsec} x + \arctan x}\left(\dfrac{1}{x\sqrt{x^2-1}} + \dfrac{1}{1+x^2}\right)$

Ex: Find y' for $y = \tan^{-1}\left(x - \sqrt{1-x^2}\right) \rightarrow y' = \dfrac{1}{1+\left(x-\sqrt{1-x^2}\right)^2} \cdot \left(1 - \dfrac{1}{2\sqrt{1-x^2}} \cdot (-2x)\right)$

Ex: Find y' for $y = \sin^{-1}(\cos^{-1}x)$

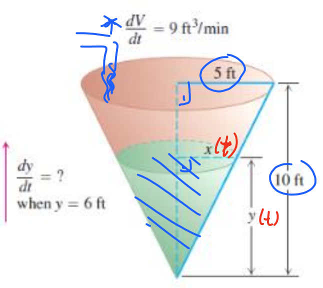
$y' = \dfrac{1}{\sqrt{1-(\arccos x)^2}} \cdot \dfrac{-1}{\sqrt{1-x^2}}$

36. $y = \cos^{-1}(e^{-t})$

$y' = \dfrac{-1}{\sqrt{1-(e^{-t})^2}} \cdot e^{-t} \cdot (-1)$ //

3.10 Related Rates

rate of change of Volume $= \dfrac{dV}{dt} = 9\ ft^3/min$



EXAMPLE 1 Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

particular moment

rate of change of y $= \dfrac{dy}{dt}\Big|_{t} ?$ at the moment $y = 6$ ft.

Solution Figure 3.43 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

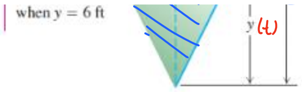


FIGURE 3.43 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

V = volume (ft³) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

Volume of a cone with r, h

$V = \frac{1}{3}\pi r^2 h$

Similarity of triangles

$\frac{x}{5} = \frac{y}{10} \Rightarrow x = \frac{y}{2}$

- Given

$\frac{dV}{dt} = 9\ \text{ft}^3/\text{min}$

Want to find

$\left.\frac{dy}{dt}\right| = ?$ at the moment $y = 6\text{ft}$

- find the relating eqn:

$V = \frac{1}{3}\pi x^2 y$

$= \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 \cdot y$

$V = \frac{1}{12}\pi y^3$

- $\frac{d}{dt}\left(\frac{dV}{dt} = \frac{1}{12}\pi\, 3y^2 \frac{dy}{dt}\right)\frac{d}{dt}$

$\frac{1}{4}$

$9 = \frac{1}{4}\pi(6)^2\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\pi}\ \text{ft/min}.$

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

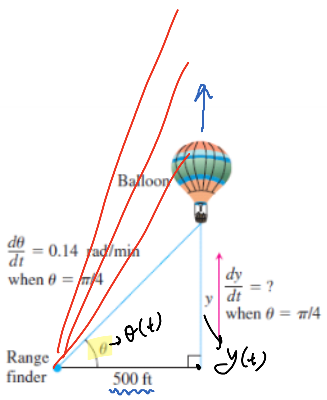


FIGURE 3.44 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

$\left.\frac{d\theta}{dt}\right| = 0.14$ rad/min at the moment when $\theta = \pi/4$

$\left.\frac{dy}{dt}\right|$ at the moment when $\theta = \pi/4$

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.44). The variables in the picture are

   θ = the angle in radians the range finder makes with the ground.

   y = the height in feet of the balloon.

- Given that

$\left.\frac{d\theta}{dt}\right| = 0.14$ rad/min at the moment when $\theta = \pi/4$

Want to find

$\left.\frac{dy}{dt}\right| = ?$ at the moment when $\theta = \pi/4$

- relating eqn: $\tan\theta = \frac{y}{}$

when $\theta = \pi/4$ ... when $\theta = \pi/4$

- relating eqn: $\tan\theta = \dfrac{y}{500}$ $\quad \dfrac{d}{dt}$

- take time derivative of both side:

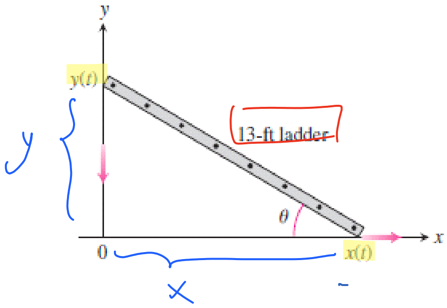
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt}$$

$$\sec^2\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = 2 \qquad \left(\cos^2\left(\tfrac{\pi}{4}\right) = \tfrac{1}{2}\right)$$

- at the moment $\theta = \frac{\pi}{4}$

$$\Rightarrow \underbrace{\sec^2\left(\frac{\pi}{4}\right)}_{2} \cdot 0{,}14 = \frac{1}{500} \cdot \frac{dy}{dt}$$

$$2 \cdot 0{,}14 = \frac{1}{500} \cdot \frac{dy}{dt} \Rightarrow \left.\frac{dy}{dt}\right|_{\theta = \pi/4} = 140\ \text{ft/min}$$



23. A sliding ladder A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. $\rightarrow \left.\dfrac{dx}{dt}\right|_{x=12} = 5\ \text{ft/sec}$

a. How fast is the top of the ladder sliding down the wall then?

b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

c. At what rate is the angle θ between the ladder and the ground changing then?

a) - Given that $\left.\dfrac{dx}{dt}\right|_{x=12} = 5\ \text{ft/sec}$ $\qquad$ Want to find $\left.\dfrac{dy}{dt}\right|_{x=12} = ?$

- relating eqn: $x^2 + y^2 = 13^2$ $\quad \dfrac{d}{dt}$

- take $\dfrac{d}{dt}$ of both sides:

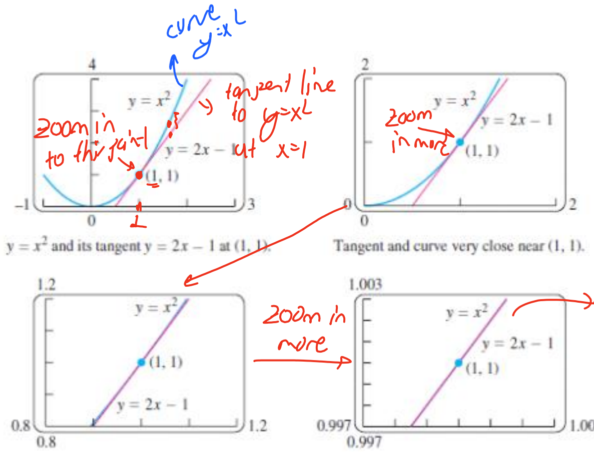
$$2x \cdot \frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$$

at the moment when $x = 12$ $\quad 12^2 + y^2 = 13^2 \rightarrow y = 5$

$$\rightarrow 2 \cdot 12 \cdot 5 + 2 \cdot 5\,\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -12\ \text{ft/sec}$$

y is decreasing.

3.11 Linearization and Differentials



$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).

Tangent and curve very close throughout entire x-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

at the tangency point the function and the tangent line are so close to each other, almost the same.

the value of the function and the value of ...

FIGURE 3.49 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its entire x-interval shown.

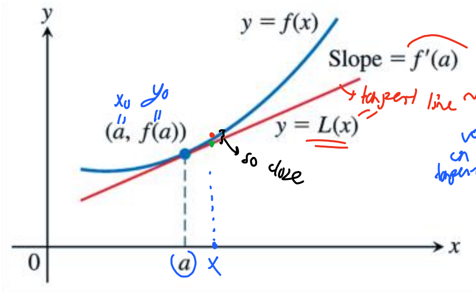
screen cannot distinguish tangent from curve on this x-interval.

FIGURE 3.49 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

...values of the function and the value of the tangent line are so close!

Figure 3.52

The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.



Eqn of tangent line: $y - f(a) = f'(a)(x-a)$

values on tangent line $y = L(x) = f(a) + f'(a)(x-a)$

Linearization fnc. of $f(x)$ at $x=a$

$L(x) \approx f(x)$

use linearization fnc to approximate $f(x)$

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$ → importance of linearization

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = \underset{a}{0}$ (Figure 3.51).

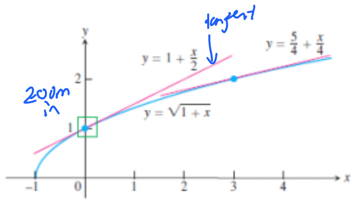


FIGURE 3.51 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.52 shows a magnified view of the small window about 1 on the y-axis.

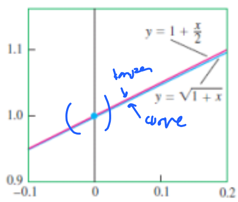


FIGURE 3.52 Magnified view of the window in Figure 3.51.

$L(x) = f(a) + f'(a)(x-a)$

$= f(0) + f'(0)(x-0)$

$L(x) = 1 + \frac{1}{2} \cdot x$

$L(x) = 1 + \frac{x}{2}$

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2\sqrt{1+x}} \rightsquigarrow f'(0) = \frac{1}{2}$

$f(0) = 1$

$f(x) \approx L(x)$

$f(x) = \sqrt{1 + x} \approx 1 + (x/2)$

Approximation $L(x)$	$\approx$	True value $f(x)$
$f(0.2) = \sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$		1.095445
$f(0.05) = \sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$		1.024695

	$L(x)$	$\approx$	$f(x)$
$f(0.2) = \sqrt{1.2} \approx 1 + \frac{0.2}{2}$	$= 1.10$	$\longleftrightarrow$	1.095445
$f(0.05) = \sqrt{1.05} \approx 1 + \frac{0.05}{2}$	$= 1.025$	$\longleftrightarrow$	1.024695
$f(0.005)\ \sqrt{1.005} \approx 1 + \frac{0.005}{2}$	$= 1.00250$	$\longleftrightarrow$	1.002497

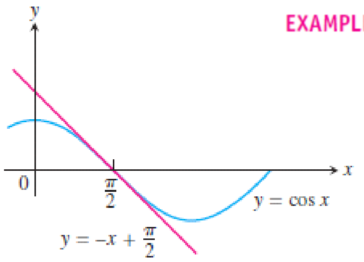


FIGURE 3.53 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.53).

$L(x) = f(\pi/2) + f'(\pi/2)(x - \frac{\pi}{2})$ $\qquad f(x) = \cos x \rightsquigarrow f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x$

$L(x) = 0 - 1\,(x - \pi/2)$ $\qquad f'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2}) = -1$

$\Downarrow$

$L(x) = -x + \frac{\pi}{2} \approx f(x) = \cos x$

Ex: Find approximately $(0,9)^{0,9}$ — choose the closest integer

$f(x) = x^x \rightsquigarrow$ find linearization of $f(x)$ at $x = 1$.

$f(1) = 1$ $\qquad L(x) = f(1) + f'(1)(x-1)$

$f'(x) = x^x(\ln x + 1)$ $\qquad = 1 + 1\,(x-1)$

$f'(1) = 1$

$\Downarrow$

$L(x) = 1 + x - 1 = x$

$f(x) \approx L(x) = x$

$(0,9)^{0,9} = f(0,9) \approx L(0,9) = 0,9$

Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

$\frac{dy}{dx} = f'(x)$

$dy = f'(x).dx$

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$. $\longrightarrow dy = \underbrace{5x^4 + 37}_{f'(x)} . dx$

(b) Find the value of dy when $x = 1$ and $dx = 0.2$. $\quad dy = (5.(1) + 37).\,0,2 = 8,4$

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

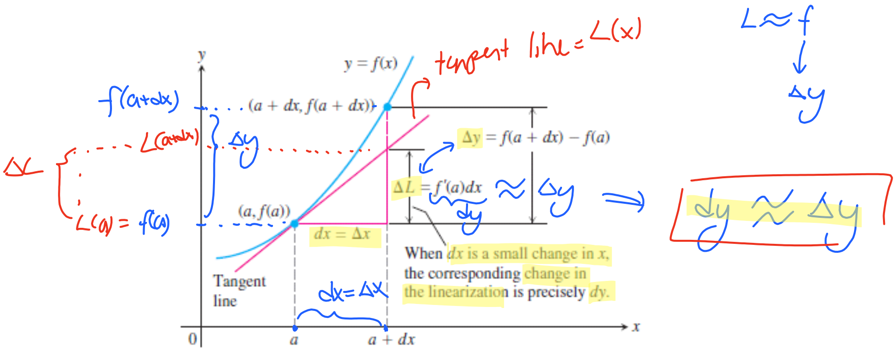


FIGURE 3.54 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 3.54 that Δy is approximately equal to the differential dy. Since

$$f(a + dx) = f(a) + \Delta y, \qquad \Delta x = dx$$

$\Delta y = f(a+dx) - f(a)$

$$\boxed{f(a + dx) \approx f(a) + dy}$$

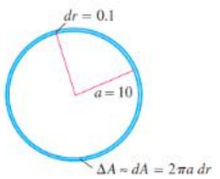


FIGURE 3.55 When dr is small compared with a, the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.55). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

$dr = 0.1$

$A(r) = \pi r^2$

$A'(r) = 2\pi r$

$dA = 2\pi r \cdot dr$

Solution Since $A = \pi r^2$, the estimated increase is

differential $\rightarrow$ $dA = A'(a)\, dr = 2\pi a\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$A(10 + 0.1) \approx A(10) + 2\pi$$
$$= \pi(10)^2 + 2\pi = 102\pi. \rightarrow \text{approximated value}$$

$A(10.1) = 102.01\pi$

$A(10.1) \approx 102\pi$ — there is very small error.

The area of a circle of radius 10.1 m is approximately 102π m².

The true area is

$A(r) = \pi r^2 \rightarrow r = 10.1 \leftarrow$ enlarged radius

$$A(10.1) = \pi(10.1)^2$$
$$= 102.01\pi \text{ m}^2. \rightarrow \text{exact value of area of the enlarged circle.}$$

The error in our estimate is 0.01π m², which is the difference $\Delta A - dA$.