Lecture_Template

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{y \to x} \frac{f(x) - f(y)}{x - y}$$

$$\cancel{(f \cdot g)' = f' \cdot g'}$$

$$(f \pm g)' = f' \pm g' \qquad (c \cdot f)' = c f' \qquad (f \cdot g)' = f' \cdot g + f \cdot g'$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$(x^6)' = 6x^5 \qquad (x^n)' = n \cdot x^{n-1}$$

$$(e^x)' = e^x \qquad (\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x \qquad (\sec x)' = \sec x \cdot \tan x$$

$$(\csc x)' = -\csc x \cdot \cot x$$

$$(a^x)' = a^x \cdot \ln a \qquad (3^x)' = 3^x \cdot \ln 3 \qquad (\ln x)' = \frac{1}{x}$$

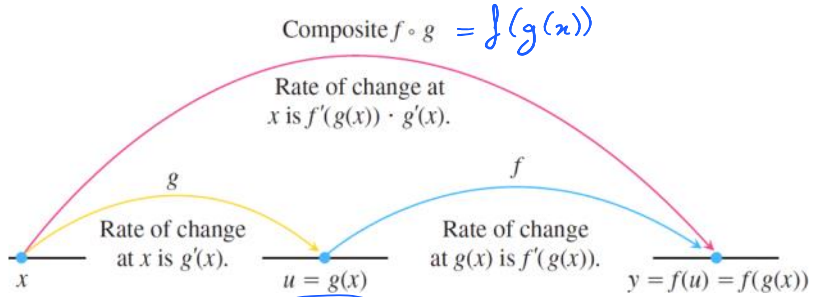
3.6 The Chain Rule

$$(\cos x)' = -\sin x \qquad (x^3 + x)' = 3x^2 + 1$$

$$(\cos(x^3 + x))'$$

$$f(x) = \cos x \qquad g(x) = x^3 + x \qquad f \circ g(x) = f(g(x)) = f(x^3 + x)$$

$$= \cos(x^3 + x)$$



$$x \longmapsto x^3 + x = u \longmapsto y = \cos u = \cos(x^3 + x)$$

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Examples: Find y' if

a) $y = \sin(x^2 + x) \Rightarrow y' =$

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b) $y = \sin(\cos x) \Rightarrow y' =$

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c) $y = \sin^2 \Rightarrow y' =$

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d) $y = \sin^2(\cos(x^3)) \Rightarrow y' =$

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e) $y = \tan^3(\sec x) \Rightarrow y =$

$= \underbrace{(\tan(\sec x))^3}$

$(A^3)' = 3A^2 \cdot A'$

---

f) $u = e^{x^2} - e^{\cos(x^5)} \Rightarrow u' =$

f) $y = e^{x^2} - e^{\cos(x^5)} \Rightarrow y' =$

$\Rightarrow (e^A)' = e^A \cdot A'$

g) $y = e^{(x^3+\cos x)^{\pi} \cdot x} \Rightarrow e^{f \cdot g} = e^{f \cdot g} \cdot (f' \cdot g + f \cdot g')$

$y' = e^{(x^3+\cos x)^{\pi} \cdot x}$

Ex: Find $\boxed{f'(x) \text{ at } x=0}$ if

$\rightarrow f(u) = \frac{2u}{u^2+1}$ and $u = g(x) = 10x^2 + x + 1$ $\quad f(x) = \frac{2(10x^2+x+1)}{(10x^2+x+1)^2+1}$

$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ $\quad$ when $x=0 \Rightarrow u = ?$

$=$

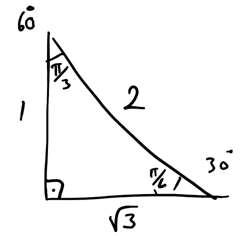
$\left.\frac{df}{dx}\right|_{x=0} =$

Ex: If $\boxed{r = \sin(f(t))}$, $f(0) = \frac{\pi}{3}$ and $f'(0) = 4$ then find $\boxed{\frac{dr}{dt} \text{ at } t=0.}$?

$\frac{dr}{dt} = \cos(f(t)) \cdot f'(t)$ $\quad \left.\frac{dr}{dt}\right|_{t=0} =$

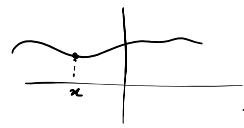
60°

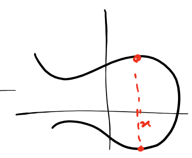
$t=0$



3.7 Implicit Differentiation

$f(x) = y = x^3 + x + 5 \longrightarrow$ explicit equation.

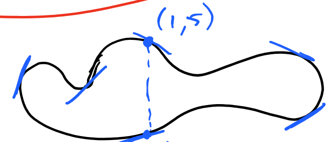




curves

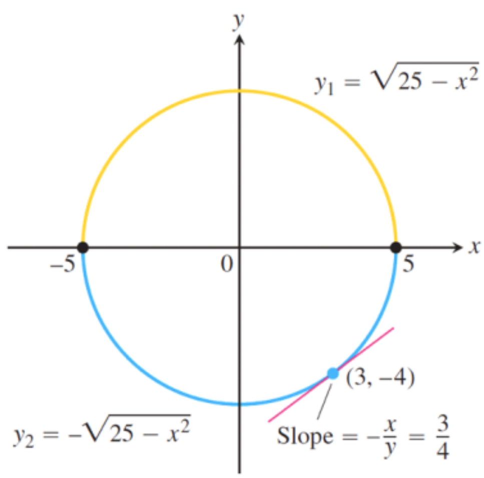
$xy = x^3y^2 + x + 5 \longrightarrow$ $\boxed{\text{implicit equation}}$

$xy - x^3y^2 = x + 5 \qquad y(x - x^3y) = x + 5$



EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

I st way : $y^2 = 25 - x^2 \implies y = \pm\sqrt{25 - x^2}$



$$y = -\sqrt{25 - x^2} = -(25 - x^2)^{1/2}$$

$$y' = -\frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x)$$

$$y' = \frac{x}{\sqrt{25 - x^2}}$$

$$y'\Big|_{x=3} = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$$

II nd way : Implicit differentiation

$y = f(x)$.

$x^2 + y^2 = 25$.

x: variable

y: depends on x.

Ex: Find $\frac{dy}{dx} = y'$ for the following curves.

a) $2xy + y^2 = x + y$

$2xy + y^2 = x + y$ $\quad\quad \frac{dx}{dy} = ?$ $\quad\quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

b) $x^2(x - y)^2 = x^2 - y^2 \implies \frac{dy}{dx} = ?$

c) $x^3 = \frac{2x - y}{x + 3y}$

d) $x^4 + \sin y = x^3y^2$

e) $e^{2x} = \sin(x + 3y)$

Ex: Find $dr/d\theta$

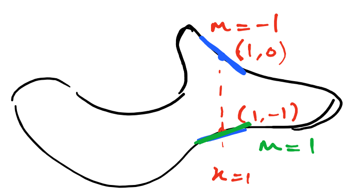
a) $\theta^{1/2} + \underline{r^{1/2}} = 1$

b) $\left(\cos r + \cot \theta = e^{r\theta}\right)$

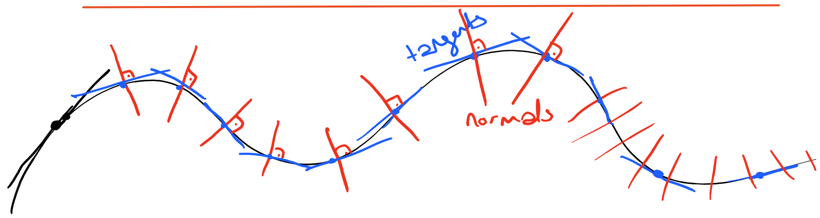
* Ex: If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

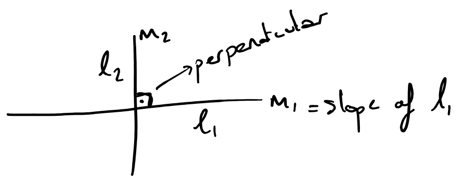
Ex: find the slope of the curve at the given point

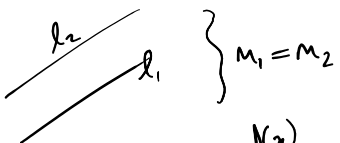
$$(x^2 + y^2)^2 = (x - y)^2 \quad \text{at} \quad (1, 0) \text{ and } (1, -1)$$

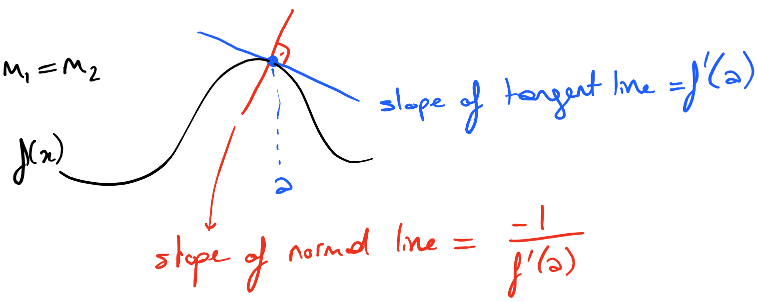


Slopes, Tangents, and Normals









Ex: **(a)** tangent and **(b)** normal to the curve

Ex: $x \cdot \sin \underline{\underline{2y}} = y \cdot \cos 2x, \quad (\pi/4, \pi/2) \quad \Rightarrow y' = ? \quad \frac{dy}{dx}$

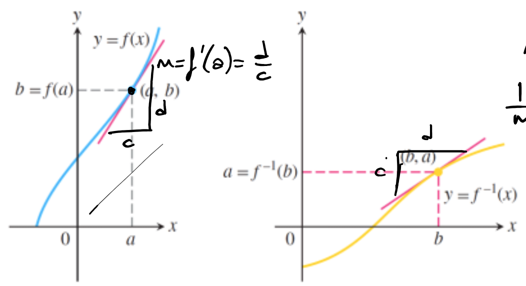
3.8 Derivatives of Inverse Functions and Logarithms

$$\frac{d\left(f^{-1}(x)\right)}{dx} = \left(f^{-1}(x)\right)' = ?$$

$$\left(f^{-1}(x)\right)'\Big|_{x=b}$$

$$\parallel$$

$$\frac{1}{m} = \frac{1}{f'(a)} = \frac{c}{d}$$



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

$$\left(f^{-1}(x)\right)'\Big|_{x=b} = \frac{1}{f'(a)} = \frac{1}{f'\left(f^{-1}(b)\right)}$$

THEOREM 3—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \qquad (1)$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Ex:

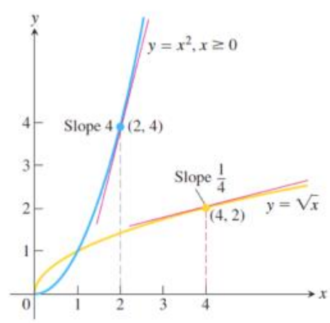


FIGURE 3.36 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point $(4, 2)$ is the reciprocal of the derivative of $f(x) = x^2$ at $(2, 4)$ (Example 1).

$y = x^2$

$y = \sqrt{x} =$

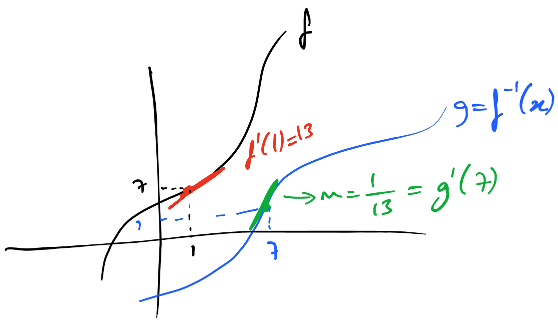
$y'\Big|_{x=4} =$

Ex: Given that the function $f(x) = x^5 + 2x^3 + 2x + 2$ has an inverse function $g(x)$, compute $g'(7)$.

$$g'(7) = \left(f^{-1}(x)\right)'\Big| = \frac{1}{\ \ } = \frac{1}{\ \ } = \frac{1}{1\ldots}$$

function $g(x)$, compute $g'(7)$.

$$g'(7) = \left(f^{-1}(x)\right)'\Big|_{x=7} = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(1)} = \frac{1}{13}$$



Derivative of the Natural Logarithm Function

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \qquad x \neq 0$$

The Derivatives of a^u and $\log_a u$

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}.$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}. \qquad (7)$$

$\log_a u = \frac{\ln u}{\ln a}$ Ex: $(\log_5 x)' =$

Ex: $(\log_{x^2} x^3 + 5)' =$

②

②Example : Find $y' = \frac{dy}{dx}$ for the following functions.

③ Examp : Find $y' = \frac{dy}{dx}$ for the following functions.

① $y = \ln x^3$

② $y = (\ln x)^3$

③ $y = \ln(\ln(\ln x))$

④ $y = \ln \dfrac{1}{x\sqrt{x+1}}$

⑤ $y = \ln \sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}} =$

⑥ $y = \log_5 \sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$

Logarithmic Differentiation

Differentiate $f(x) = x^x,\ x > 0$.

$(x^2)' = 2x$

$y = x^x$

$(2^x)' = 2^x \ln 2$

Take ln of both sides

Take derivative of both sides.

Examples: Find y'.

1 $x^y = y^x$

$y = x^{\sin x}$

$y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

k ,

Q2

Q3