Lecture

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{y\to x}\frac{f(x)-f(y)}{x-y}$$

~~$(f\cdot g)' \neq f'\cdot g'$~~

$(f\pm g)'=f'\pm g'$ $\quad (c\cdot f)'=c\,f'$ $\quad (f\cdot g)'=f'\cdot g+f\cdot g'$

$$\left(\frac{f}{g}\right)'=\frac{f'\cdot g-f\cdot g'}{g^2}$$

$(x^6)'=6x^5$ $\quad (x^n)'=n\cdot x^{n-1}$

$(e^x)'=e^x$ $\quad (\sin x)'=\cos x$ $\quad (\cos x)'=-\sin x$

$(\tan x)'=\sec^2 x$ $\quad (\cot x)'=-\csc^2 x$ $\quad (\sec x)'=\sec x\cdot\tan x$

$(\csc x)'=-\csc x\cdot\cot x$

➜ $(a^x)'=a^x\cdot\ln a$ $\quad (3^x)'=3^x\cdot\ln 3$ $\quad (\ln x)'=\frac{1}{x}$

3.6 The Chain Rule

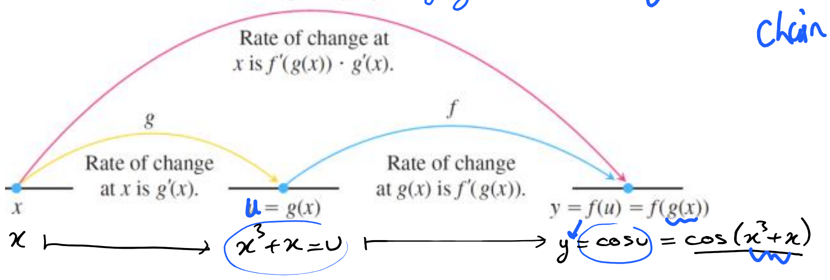
$(\cos x)'=-\sin x$ $\quad (x^3+x)'=3x^2+1$

$\big(\cos(\underbrace{x^3+x}_{g})\big)'$

$f(x)=\cos x$ $\quad g(x)=x^3+x$ $\quad f\circ g(x)=f(g(x))=f(x^3+x)$

$=\cos(x^3+x)$

Composite $f\circ g$ $=f(g(x))$ ⇝ $[f(g(x))]'=?$ → Chain Rule.



THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f\circ g)(x)=f(g(x))$ is differentiable at x, and

$$(f\circ g)'(x)=f'(g(x))\cdot g'(x).$$

In Leibniz's notation, if $y=f(u)$ and $u=g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx},$$

where dy/du is evaluated at $u=g(x)$.

$y=f(g(x))$, $u=g(x)$ $\quad f(g(x))'=f'(g(x))\cdot g'(x)$

$(f\circ g)'=(f(g(x)))'=f'(g(x))\cdot g'(x)$ ← Chain Rule

Examples: Find y' if

Examples: Find y' if

a) $y = \sin(x^2+x) \Rightarrow y' = \cos(x^2+x) \cdot (x^2+x)'$

($f = \sin$, $g(x) = x^2+x$; $\cos(x^2+x) = f'(g(x))$, $(x^2+x)' = 2x+1 = g'(x)$)

Leibniz Rule: $y = f(u)$, $u = x^2+x$

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$u = x^2+x$, $f(u) = \sin u$

$$= \cos u \cdot (2x+1)$$
$$= \cos(x^2+x) \cdot (2x+1)$$

---

b) $y = \sin(\cos x) \Rightarrow y' = \cos(\cos x) \cdot (\cos x)'$ (derivative)

$$= \cos(\cos x) \cdot (-\sin x)$$

---

c) $y = \sin^2 x \Rightarrow y' = 2(\sin x)^1 \cdot (\sin x)'$

$y = (\sin x)^2$ ← handle this first

$$= 2\sin x \cdot \cos x$$

---

d) $y = \sin^2(\cos(x^3)) \Rightarrow y' = 2(\sin(\cos(x^3)))^1 \cdot (\sin(\cos(x^3)))'$

$= \sin(\cos(x^3))^2$

$(\sin(\cos(x^3)))' = \cos(\cos(x^3)) \cdot (\cos(x^3))'$, $(\cos(x^3))' = -\sin(x^3) \cdot (x^3)'$, $(x^3)' = 3x^2$

$$= 2(\sin(\cos(x^3))) \cdot \cos(\cos(x^3)) \cdot -\sin(x^3) \cdot 3x^2 \;//$$

---

e) $y = \tan^3(\sec x) \Rightarrow y' = 3(\tan(\sec x))^2 \cdot (\tan(\sec x))'$

$= (\tan(\sec x))^3$

$(A^3)' = 3A^2 \cdot A'$

$(\tan(\sec x))' = \sec^2(\sec x) \cdot (\sec x)'$, $(\sec x)' = \sec x \cdot \tan x$

$$= 3(\tan(\sec x))^2 \cdot \sec^2(\sec x) \cdot \sec x \cdot \tan x \;//$$

f) $y = e^{x^2} - e^{\cos(x^5)} \Rightarrow y' = e^{x^2}.2x - e^{\cos(x^5)} \cdot (\cos(x^5))'$

$-\sin(x^5).5x^4$

Recall) $(e^{f(x)})' = e^{f(x)}.f'(x)$

$\Rightarrow (e^A)' = e^A.A'$ $\quad = e^{x^2}.2x - e^{\cos(x^5)} \cdot -\sin(x^5).5x^4$

g) $y = e^{(x^3+\cos x)^{\pi} \cdot x}$ $\Rightarrow$ $e^{f \cdot g} = e^{f \cdot g}.(f'.g + f.g')$ product rule.

$y' = e^{(x^3+\cos x)^{\pi} \cdot x} \cdot [(x^3+\cos x)^{\pi} \cdot x]'$

$y' = e^{(x^3+\cos x)^{\pi} \cdot x} \cdot [\pi(x^3+\cos x)^{\pi-1}.(3x^2-\sin x) \cdot x + (x^3+\cos x)^{\pi}.1]$

(f', g, f, g') derivative of power.

Ex: Find $f'(x)$ at $x=0$ if

$\rightarrow f(u) = \frac{2u}{u^2+1}$ and $u = g(x) = 10x^2+x+1$ $\qquad f(x) = \frac{2(10x^2+x+1)}{(10x^2+x+1)^2+1} \rightsquigarrow f'|_{x=0} = ?$

Leibniz Rule: $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ $\qquad$ when $x=0 \Rightarrow u = ?$

$= \frac{2.(u^2+1) - 2u.2u}{(u^2+1)^2} \cdot 20x+1$ $\qquad$ Recall) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

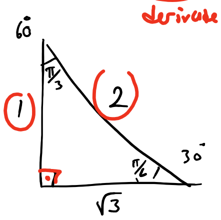
$\left.\frac{df}{dx}\right|_{x=0 \,(u=1)} = \frac{2.2-2.2}{(4+1)^2}.(20.0+1) = 0$

Ex: If $r = \sin(f(t))$, $f(0) = \frac{\pi}{3}$ and $f'(0) = 4$ then find $\frac{dr}{dt}$ at $t=0$?

Chain Rule

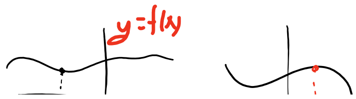
$\frac{dr}{dt} = \cos(f(t)) \cdot f'(t)$ (derivative.) $\qquad \left.\frac{dr}{dt}\right|_{t=0} = \cos(f(0)).f'(0)$ ($\pi/3$, 4)

$= \cos(\pi/3).4 = 2$ ($\cos(\pi/3) = 1/2$)



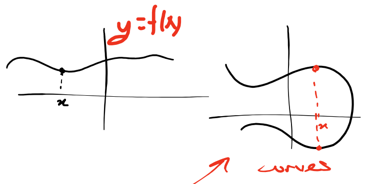
3.7 Implicit Differentiation

$y(x) = x^3 + x + 5 \longrightarrow$ explicit equation.



3.7 Implicit Differentiation

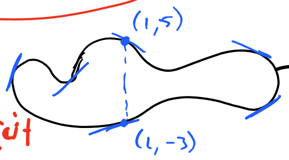
$f(x) = y = x^3 + x + 5$ → explicit equation. → $\frac{dy}{dx} = \checkmark$



$xy = x^3y^2 + x + 5$ → implicit equation → $\frac{dy}{dx} = ?$ ⇓ by implicit diff.

$xy - x^3y^2 = x + 5$ → $y(x - x^3y) = x + 5$

$y = f(x) = ?$



→ curve defined by an implicit eqn ⇝ $\frac{dy}{dx} = \checkmark$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Ist way: $y^2 = 25 - x^2 \implies y = \pm\sqrt{25 - x^2}$

$y = -\sqrt{25 - x^2} = -(25 - x^2)^{1/2}$

derivative

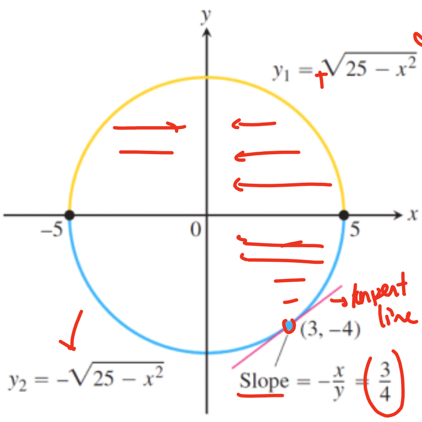
$y' = -\frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x)$

$y' = \frac{x}{\sqrt{25 - x^2}}$

$y'\Big|_{x=3} = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$ ← slope of tangent line to curve $y = -\sqrt{25 - x^2}$ at $x = 3$

Recall

m = slope of the tangent line at $x=3$ = $\frac{dy}{dx}\Big|_{x=3}$



IInd way: Implicit differentiation

$x^2 + y^2 = 25$ ← implicit eqn ($y = f(x)$)

x: variable; y: depends on x. ⇝ $y(x)$

$\frac{d}{dx}\left(x^2 + y^2 = 25\right) \frac{d}{dx}$

$2x + 2y \cdot y' = 0$

$y' = \frac{-2x}{2y}$

$y'\Big|_{x=3,\ y=-4} = \frac{-3}{-4} = \frac{3}{4}$

Remark,

$(y(x))^2 ⇝ \left[(y(x))^2\right]' = 2\, y(x) \cdot y'(x)$

Ex: Find $\frac{dy}{dx} = y'$ for the following curves.

a) $2xy + y^2 = x + y$ ⇝ implicit eqn ⇝ $y(x) = f(x)$ = not possible. $\frac{dy}{dx} = ?$

(product) $\frac{d}{dx}$

$2y + 2x \cdot y' + 2y \cdot y' = 1 + y'$

$(2x + 2y - 1)\, y' = 1 - 2y$

$1 - 2y$

Remark

$(2x \cdot y)' = 2 \cdot y + 2x \cdot y'$ (f = 2x, g = y; $f'g + fg'$) ← product rule

$(2x+2y-1)\,y' = 1-2y$

$$\frac{dy}{dx} = y' = \frac{1-2y}{2x+2y-1}$$

compare : $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

- $2xy + y^2 = x + y$ $\qquad \dfrac{dx}{dy} = ?$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Consider $x = x(y)$

$\dfrac{d}{dy}$ $\rightarrow$ $2 \cdot \dfrac{dx}{dy} \cdot y + 2x \cdot 1 + 2y = \dfrac{dx}{dy} + 1$

$(2y - 1)\dfrac{dx}{dy} = 1 - 2x - 2y$

$$\frac{dx}{dy} = \frac{1-2x-2y}{2y-1}$$

b) $x^2(x-y)^2 = x^2 - y^2 \implies \dfrac{dy}{dx} = ?$

c) $x^3 = \dfrac{2x-y}{x+3y} \implies \dfrac{dy}{dx} = ?$

d) $x^4 + \sin y = x^3y^2$

e) $e^{2x} = \sin(x+3y)$ $\rightsquigarrow$ implicit eqn $\rightsquigarrow$ $\dfrac{dy}{dx} = ?$

Implicit differentiation

Recall $\left[e^{f(x)}\right]' = e^{f(x)} \cdot f'(x)$

$\dfrac{d}{dx}$ $\downarrow$

$e^{2x} \cdot 2 = \cos(x+3y) \cdot (1+3y')$

$\Rightarrow 1 + 3y' = \dfrac{e^{2x} \cdot 2}{\cos(x+3y)}$

$$y' = \frac{2e^{2x} - \cos(x+3y)}{\cos(x+3y)} \cdot \frac{1}{3}$$

Ex: Find $dr/d\theta$ $\rightarrow r(\theta)$

Ex: Find $dr/d\theta$

a) $\theta^{1/2} + r^{1/2} = 1$

$\frac{d}{d\theta}\left(\frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0\right.$

$\frac{1}{2\sqrt{r}}\frac{dr}{d\theta} = \frac{-1}{2\sqrt{\theta}}$

$\frac{dr}{d\theta} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

b) $(\cos r + \cot\theta = e^{r\theta})$

---

* Ex: If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

$1\cdot y + x\cdot\frac{dy}{dx} + 2y\cdot\frac{dy}{dx} = 0 \xrightarrow{x=0,\ y=-1} -1 + 0 + 2(-1)\frac{dy}{dx} = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{x=0,\ y=-1} = \frac{-1}{2}$

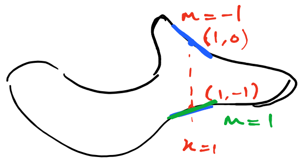
$\frac{dy}{dx} + 1\cdot\frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\cdot\frac{d^2y}{dx^2} = 0$

$x=0,\ y=-1,\ \left.\frac{dy}{dx}\right|_{x=0,y=-1} = -\frac{1}{2}$

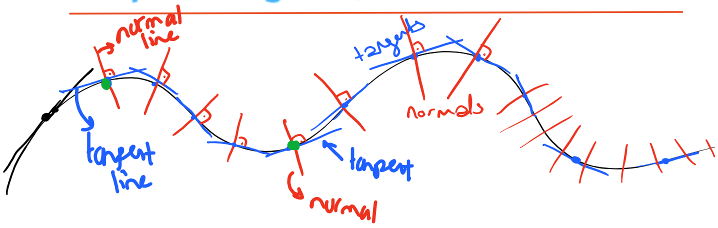
$-\frac{1}{2} + \left(-\frac{1}{2}\right) + 0\cdot + 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 2\cdot(-1)\cdot\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{4}$

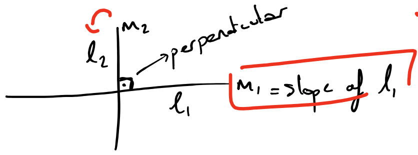
Ex: find the slope of the curve at the given point

$(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

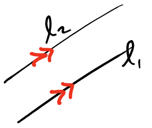


Slopes, Tangents, and Normals

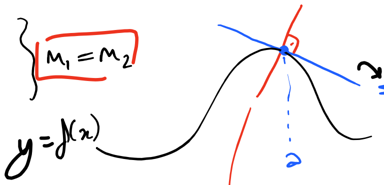




$$m_1 \cdot m_2 = -1$$



$m_1 = m_2$



slope of tangent line $= f'(a) = m$

} product $= -1$

slope of normal line $= \dfrac{-1}{f'(a)} = -\dfrac{1}{m}$

Find the eqn of

Ex: **(a)** tangent and **(b)** normal to the curve

Ex: $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$ $\Rightarrow y' = ?$ $\quad \frac{dy}{dx}$

product

the slope of tangent line $= \left.\dfrac{dy}{dx}\right|_{x=\pi/4}$

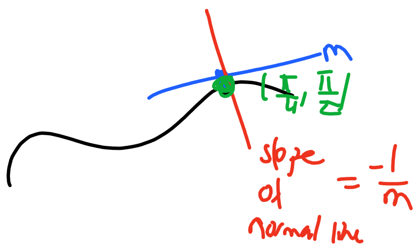
implicit diff.

$$1 \cdot \sin 2y + x \cdot \cos(2y) \cdot 2y' = y' \cdot \cos 2x + y \cdot -\sin(2x) \cdot 2$$

$x = \pi/4$, $y = \pi/2$

$$1 + \frac{\pi}{4}(-1) \cdot 2y' = \pi \Rightarrow y' = \frac{-\pi - 1}{\frac{\pi}{2} + 1} = \text{slope of the tangent}$$

- Eqn of tangent line: $m = \dfrac{-\pi - 1}{\frac{\pi}{2} + 1}$, $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (x_0, y_0)



$$y - y_0 = m(x - x_0)$$

$$y - \frac{\pi}{2} = \frac{-\pi - 1}{\frac{-\pi}{2} + 1}\left(x - \frac{\pi}{4}\right)$$

- Eqn of normal line: slope of normal line $= -\dfrac{1}{m} = \dfrac{-\frac{\pi}{2} + 1}{\pi + 1}$

point $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

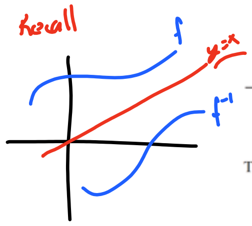
$$\left(y - \frac{\pi}{2}\right) = \frac{-\pi/2 + 1}{\pi + 1}\left(x - \frac{\pi}{4}\right)$$

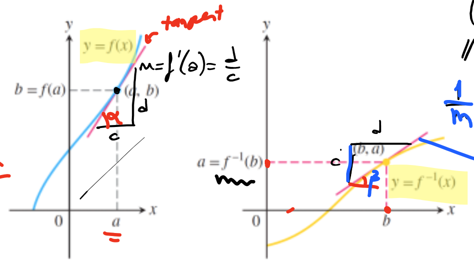
slope of normal line

3.8 Derivatives of Inverse Functions and Logarithms

$$\frac{d\left(f^{-1}(x)\right)}{dx} = \left(f^{-1}(x)\right)' = ?$$

Recall





The slopes are reciprocal: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

$$\left(f^{-1}(x)\right)'\Big|_{x=b} = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

$$\left(f^{-1}(x)\right)'\Big|_{x=b} = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

$\frac{1}{m} = \frac{1}{f'(a)} = \frac{c}{d}$

slope of tangent line to f^{-1} at $x=b$

$f(a)=b$

$f^{-1}(b)=a$

THEOREM 3—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

($f^{-1}(b) = a$)

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Ex:

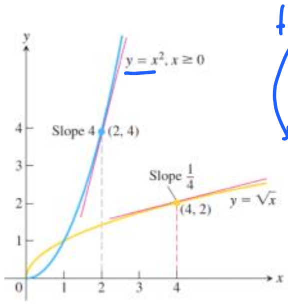


FIGURE 3.36 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 1).

$f = y = x^2 \rightsquigarrow f'(x) = 2x$

$\left.\frac{dy}{dx}\right|_{x=2} = f'(2) = 4$

$f^{-1} = y = \sqrt{x} = \quad (f^{-1})' = \frac{1}{2\sqrt{x}}$

$y'\Big|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$(f^{-1})'\Big|_4 = \frac{1}{f'(2)} = \frac{1}{4}$ (4 = b, 2 = a)

$f(a) = b$ (a = 2, b = 4)

Ex: Given that the function $f(x) = x^5 + 2x^3 + 2x + 2$ has an inverse function $g(x)$ ($= f^{-1}$), compute $g'(7) = \left(f^{-1}(7)\right)' = ?$ — not easy to compute the $f^{-1}(x) = ?$

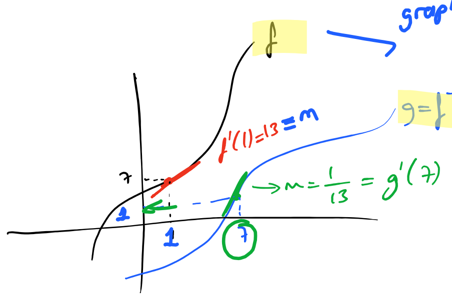
$$g'(7) = \left(f^{-1}(x)\right)'\Big|_{x=7} = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(1)} = \frac{1}{13}$$

$f^{-1}(7) = a = ?$

$f(x) = x^5 + 2x^3 + 2x + 2 \rightsquigarrow f(a) = 7$

$a^5 + 2a^3 + 2a + 2 = 7 \Rightarrow \boxed{a = 1}$

$f'(x) = 5x^4 + 6x^2 + 2 \rightsquigarrow f'(1) = 5 + 6 + 2 = 13$



graphs are symmetric w.r.t. $y=x$ line

$g=f^{-1}(x)$ = slope of green tangent line $= (f^{-1}(x))'\big|_{x=7} = \frac{1}{f'(1)} = \frac{1}{13}$ //

Derivative of the Natural Logarithm Function

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \qquad x \neq 0$$

Remarks)

$$(\ln(f(x)))' = \frac{1}{f(x)} \cdot f'(x)$$

Ex) $y = \ln(x^3) \rightsquigarrow y' = \frac{1}{x^3} \cdot 3x^2$ //

Ex) $y = (\ln x)^3 \rightsquigarrow y' = 3(\ln x)^2 \cdot \frac{1}{x}$

Ex) $y = \ln(\ln(\ln x)) \rightsquigarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$ ($\ln(\ln x)$ = $f(x)$)

The Derivatives of a^u and $\log_a u$

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

($u = u(x)$)

Remarks) General exponential fnc.

- $(a^x)' = a^x \cdot \ln a$
- $(a^{f(x)})' = a^{f(x)} \cdot \ln a \cdot f'(x)$

$a=e$: Natural exponential fnc.

- $(e^x)' = e^x$
- $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$

Ex) $(3^x)' = 3^x \cdot \ln 3$

$(3^{x^2})' = 3^{x^2} \cdot \ln 3 \cdot 2x$ ($2x = f'(x)$)

$(x^3)' = 3x^2$

$(\pi^3)' = 0$ (constant)

Warnings) $(3^x)' = \cancel{x3^{x-1}}$ wrong!

How to derivate general logarithm fnc?

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$

($u = u(x)$)

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\log_a u = \frac{\ln u}{\ln a}$$

Ex: $(\log_5 x)' = \left(\frac{\ln x}{\ln 5}\right)'$

$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$ //   ($u(x)$; $(\ln x)' = \frac{1}{x}$)

Ex: $(\log_{\cdot} x^3+5)' = \quad = \frac{1}{\cdot}(\ln x)'$

Ex: $(\log_{x^2} x^3+5)' =$

② $\left(\frac{\ln(x^3+5)}{\ln x^2}\right)'$ = $\frac{\frac{1}{x^3+5}\cdot 3x^2 \cdot \ln x^2 - \ln(x^3+5)\cdot \frac{1}{x^2}\cdot 2x}{(\ln x^2)^2}$ (quotient Rule)

$= \frac{1}{\ln 5}(\ln x)'$

$= \frac{1}{\ln 5}\cdot\frac{1}{x}$

$= \frac{1}{\ln a}\cdot\frac{1}{x}$

Logarithmic Differentiation

Differentiate $f(x) = x^x, x > 0$.

both base and the power depends on "x": $y = x^x$, $\frac{dy}{dx} = ?$

$(x^2)' = 2x$

$(2^x)' = 2^x \ln 2$

Take ln of both sides

$y = x^x$

$\ln y = \ln x^x$

$\ln y = x \cdot \ln x$ (product)

Take derivative of both sides.

$\frac{1}{y}\cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$y' = y \cdot (\ln x + 1)$

$y' = x^x (\ln x + 1)$

Recall

$(x^a)' = a x^{a-1}, \; a \in \mathbb{R}$

$(a^x)' = a^x \cdot \ln a$

$(f(x)^a)' = a\, f(x)^{a-1} \cdot f'(x)$

$(a^{f(x)})' = a^{f(x)} \cdot \ln a \cdot f'(x)$

$(f(x)^{g(x)})'$ ⇝ Logarithmic diff

$(x^x)' = \cancel{x \cdot x^{x-1}}$ Wrong!

Properties of "ln"

- $\ln(a \cdot b) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(x^n) = n \cdot \ln x$

Examples: Find y'.

① $x^y = y^x$

$y = x^{\sin x}$