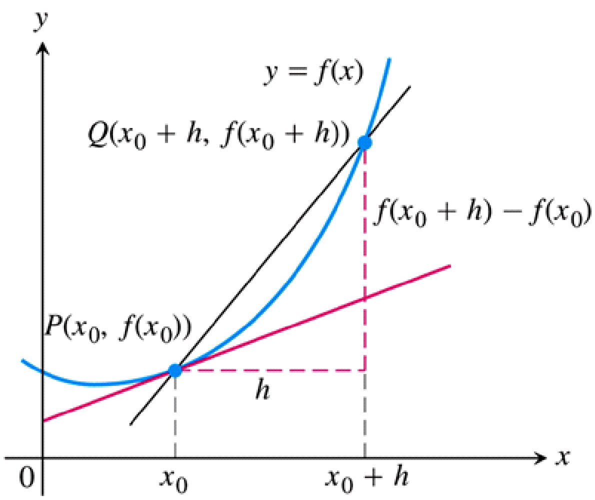
# Lecture Notes_Template

Saturday, October 17, 2020 10:30 PM

# Chapter 3. Differentiation

## 3.1 Tangents and the Derivative at a Point

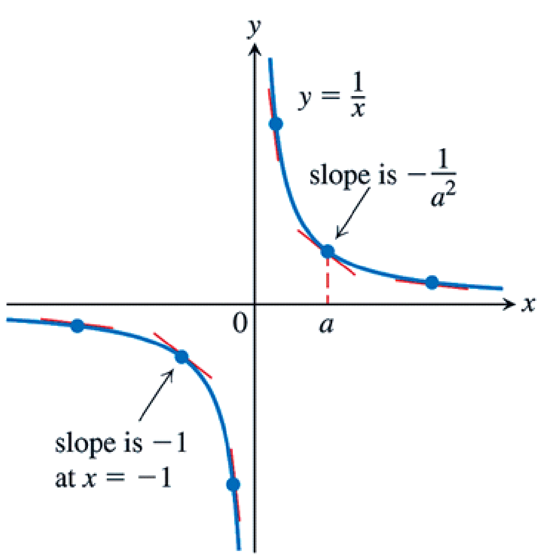



**DEFINITIONS** The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

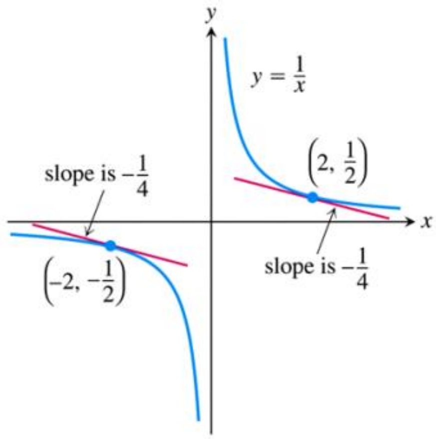
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at $P$ is the line through $P$ with this slope.

**EXAMPLE 1**




Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?




# Definition 1

The **derivative of a function *f* at a point** $x_0$, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

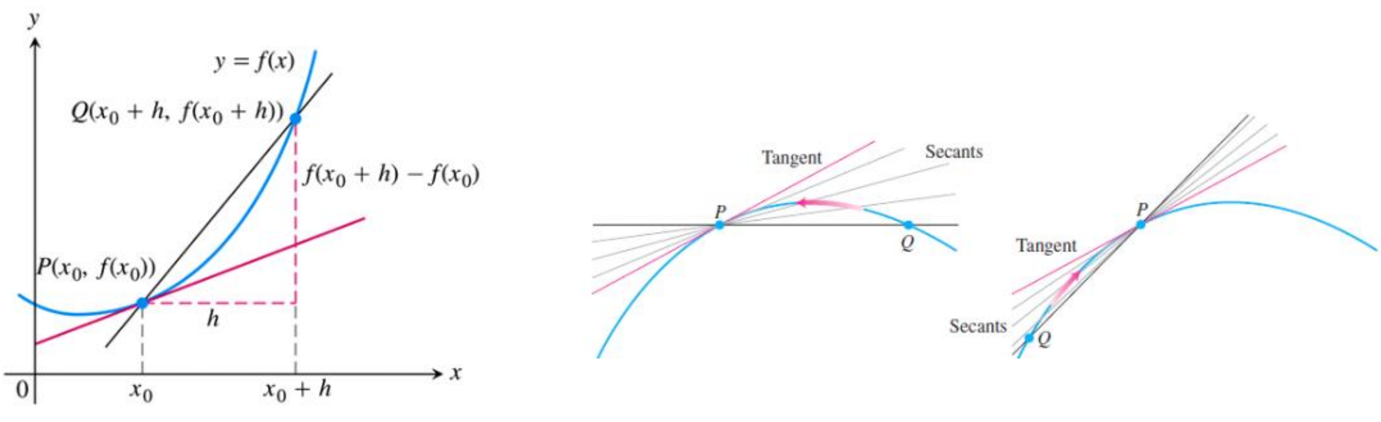



FIGURE 2.3 The tangent to the curve at $P$ is the line through $P$ whose slope is the limit of the secant slopes as $Q \to P$ from either side.

## Summary

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to *x* at the $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

**Example**

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

**11.** $f(x) = x^2 + 1, \quad (2, 5)$ **12.** $f(x) = x - 2x^2, \quad (1, -1)$

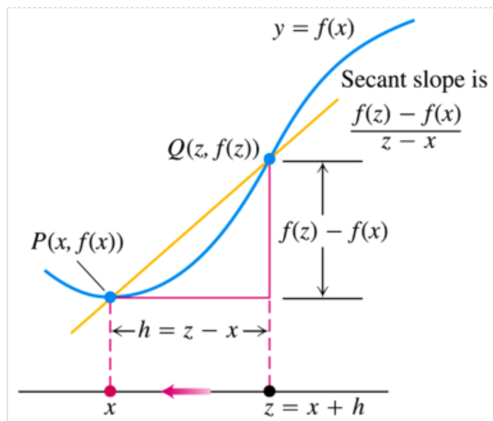
## 3.2 The Derivative as a Function

**DEFINITION** The **derivative** of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Two forms for the difference quotient.




Derivative of $f$ at $x$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

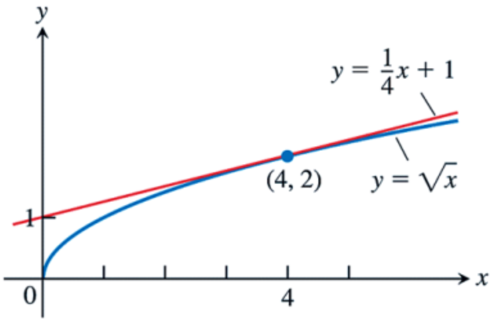
## Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

## EXAMPLE 2

**(a)** Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

**(b)** Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.




## Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is $x$ and the dependent variable is $y$. Some common alternative notations for the derivative are

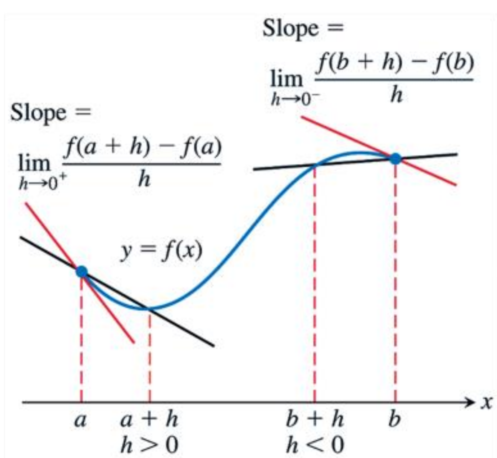
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

# Figure 3.7

Derivatives at endpoints of a closed interval are one-sided limits.



## Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it ha derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ i is differentiable on the interior $(a, b)$ and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

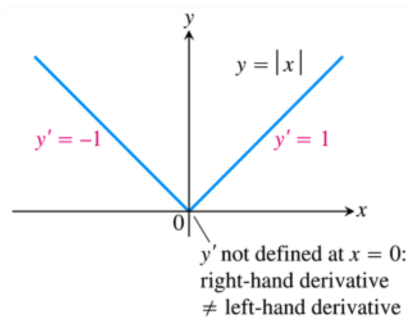
$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Figure 3.7). [redacted]

**EXAMPLE 4** Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

### Figure 3.8

The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).




(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0$$

$$= \lim_{h\to 0^+} 1 = 1$$

$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h}$$

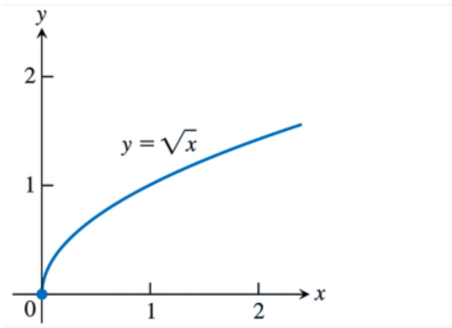
$$= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0$$

$$= \lim_{h\to 0^-} -1 = -1.$$

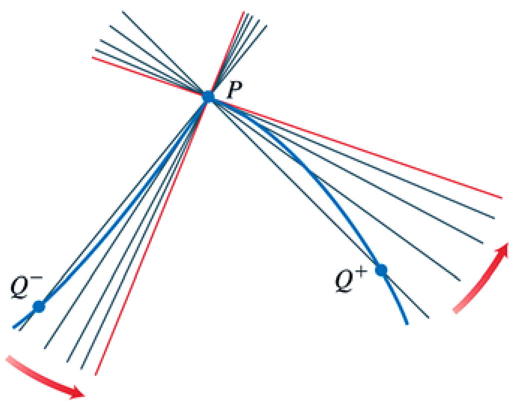
■

## Figure 3.9

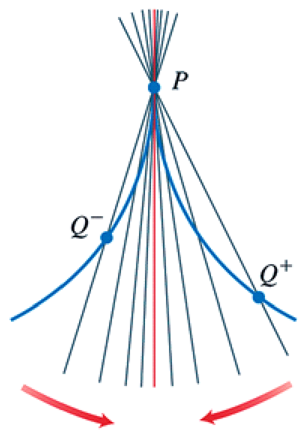
The square root function is not differentiable at $x = 0$,
where the graph of the function has a vertical tangent line.



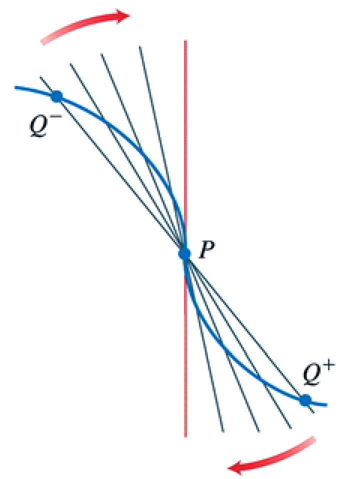

### When Does a Function *Not* Have a Derivative at a Point?



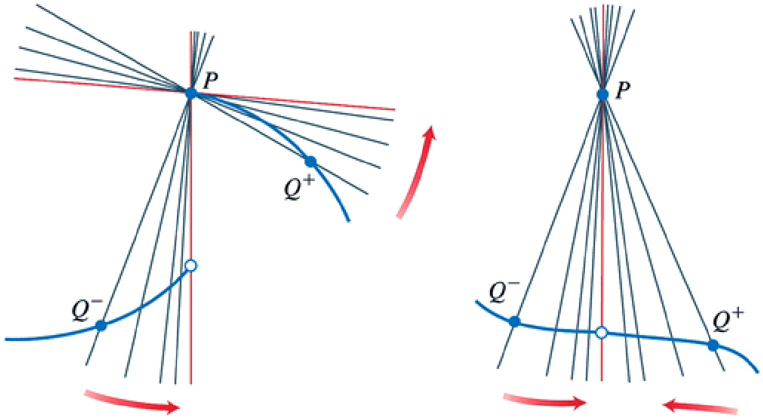

**1.** a *corner*, where the one-sided derivatives differ



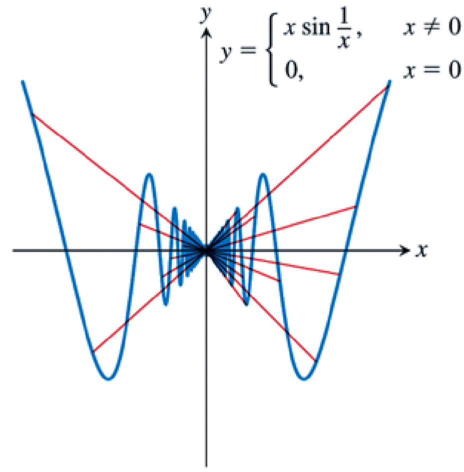

**2.** a *cusp*, where the slope of *PQ* approaches $\infty$ from one side and $-\infty$ from the other




**3.** a *vertical tangent line*, where the slope of *PQ* approaches $\infty$ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)




**4.** a *discontinuity* (two examples shown)




**5.** wild oscillation

### Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1—Differentiability Implies Continuity** If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.

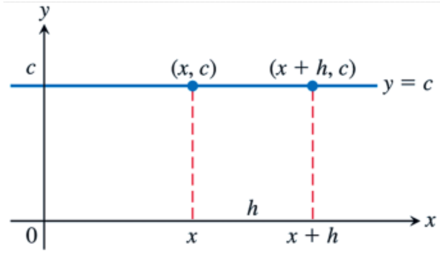
## 3.3 Differentiation Rules

### Derivative of a Constant Function

If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

The rule $\left(\frac{d}{dx}\right)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.




**Power Rule (General Version)**

If $n$ is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all $x$ where the powers $x^n$ and $x^{n-1}$ are defined.

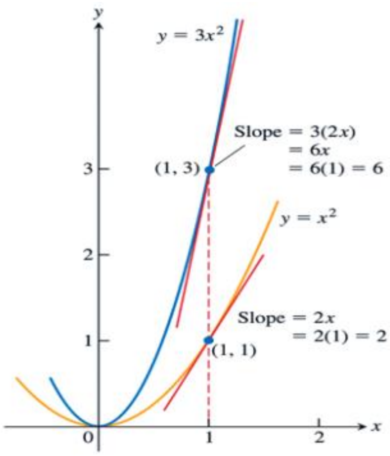
**EXAMPLE 1** Differentiate the following powers of $x$.

**(a)** $x^3$ **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

### Derivative Constant Multiple Rule

If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$




The graphs of $y = x^2$ and $y = 3x^2$.

**(b) Negative of a function**

The derivative of the negative of a differentiable function $u$ is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

■

### Derivative Sum Rule

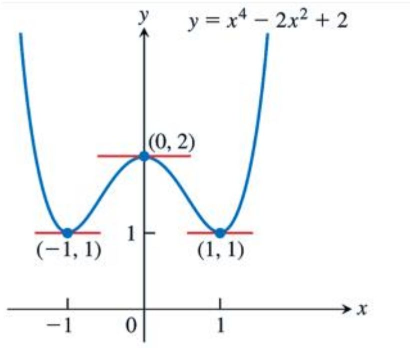
If $u$ and $v$ are differentiable functions of $x$, then their sum $u + v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

**EXAMPLE 3** Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

**EXAMPLE 4** Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

The curve in Example 4 and its horizontal tangents.




> **Derivative of the Natural Exponential Function**
>
> $$\frac{d}{dx}(e^x) = e^x$$

**EXAMPLE 5** Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

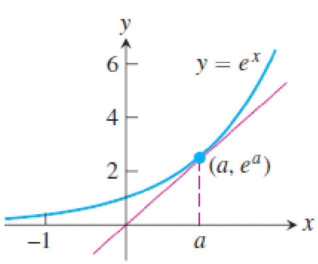



**FIGURE 3.13** The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

> **Derivative Product Rule**
>
> If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and
>
> $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

**EXAMPLE 6** Find the derivative of **(a)** $y = \frac{1}{x}\left(x^2 + e^x\right)$,

**Derivative Quotient Rule**

If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

**EXAMPLE 8** Find the derivative of **(a)** $y = \frac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

## Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If $f'$ is also differentiable, then we can differentiate $f'$ to get a new function of $x$ denoted by $f''$. So $f'' = (f')'$. The function $f''$ is called the **second derivative** of $f$ because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

$$y''' = dy''/dx = d^3y/dx^3,$$

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

**EXAMPLE 10** The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

**37.** $y = \sqrt[7]{x^2} - x^e$ **23.** $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

**34.** $y = x^{-3/5} + \pi^{3/2}$ **48.** $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

# 3.5 Derivatives of Trigonometric Functions

**The derivative of the sine function is the cosine function:**

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$ Derivative definition

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

**EXAMPLE 1** We find derivatives of the sine function involving differences, products, and quotients.

**(a)** $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x$$

**(b)** $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)\sin x \qquad \text{Product Rule}$$
$$= e^x \cos x + e^x \sin x$$
$$= e^x(\cos x + \sin x)$$

**(c)** $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

■

**The derivative of the cosine function is the negative of the sine function:**

$$\frac{d}{dx}(\cos x) = -\sin x.$$

**EXAMPLE 2** We find derivatives of the cosine function in combinations with other functions.

**(a)** $y = 5e^x + \cos x$:

**(b)** $y = \sin x \cos x$:

**(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

## Remainder

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

**The derivatives of the other trigonometric functions:**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

## Examples

**7.** $f(x) = \sin x \tan x$

**8.** $g(x) = \csc x \cot x$

**9.** $y = (\sec x + \tan x)(\sec x - \tan x)$

**10.** $y = (\sin x + \cos x) \sec x$

**17.** $f(x) = x^3 \sin x \cos x$

**22.** $s = \dfrac{\sin t}{1 - \cos t}$