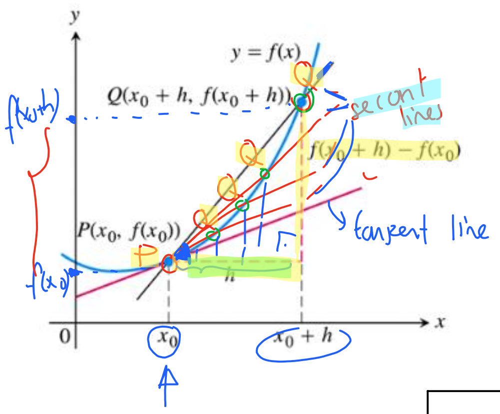
Lecture

Saturday, October 17, 2020 10:30 PM

Chapter 3. Differentiation

3.1 Tangents and the Derivative at a Point



$Q \xrightarrow{h \to 0} P \Rightarrow$ secant

secant lines $\xrightarrow{h \to 0}$ tangent line

$$\boxed{\text{slope of secant line} = \frac{f(x_0+h) - f(x_0)}{h} \xrightarrow{\lim_{h\to 0}} \text{slope of tangent line}}$$

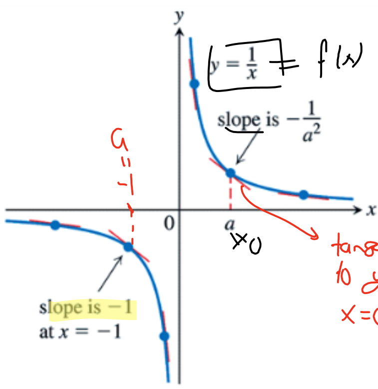
$$\boxed{m = \lim_{h \to 0} \text{slope of secant line} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \text{slope of tangent line to } y=f(x) \text{ at } x=x_0}$$

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\boxed{m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

EXAMPLE 1

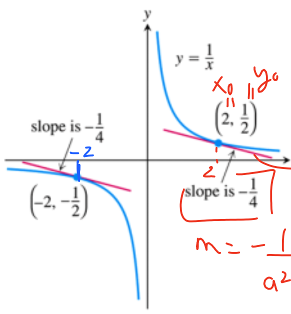


tangent line to $y = \frac{1}{x}$ at $x = a$ $\rightsquigarrow$ m (slope of tangent line)

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{\cancel{a} - \cancel{a} - \cancel{h}}{a(a+h)\cdot \cancel{h}} = \lim_{h \to 0} \frac{-1}{a(a+\cancel{h})} = \frac{-1}{a^2} = m$$

Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

$$m = \frac{-1}{a^2} = \frac{-1}{(-1)^2} = -1 \qquad a = -1$$



$$m = -\frac{1}{a^2} = -\frac{1}{2^2} = \frac{-1}{4}$$

Eqn of tangent line at $x=2$:

$$y - y_0 = m(x - x_0)$$

$$\boxed{y - \frac{1}{2} = \frac{-1}{4}(x-2)}$$

Definition 1

The derivative of a function f at a point x_0, denoted

Definition 1

The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

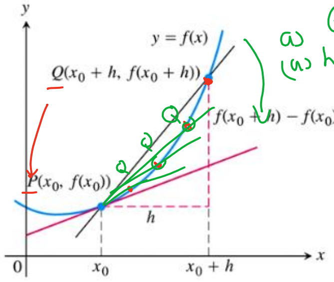
Notation: for the derivative of $f(x)$ at $x_0 = f'(x_0)$

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

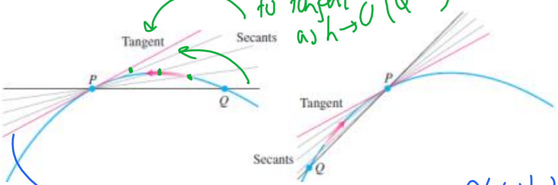
slope of the tangent line to $f(x)$ at $x = x_0$

provided this limit exists.

as $Q \to P$: secant lines $\to$ tangent line
(as $h \to 0$)



secant lines are approaching to tangent as $h \to 0$ $(Q \to P)$



slope of tangent line $= m = \lim\limits_{\substack{h \to 0 \\ (Q \to P)}} \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\text{slope of the secant line}} = f'(x_0)$

FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

Summary

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at the $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

Example

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1$, $(2, 5)$ **12.** $f(x) = x - 2x^2$, $(1, -1)$

(11) $m =$ the slope of the tangent line at $x = 2$ $= f'(2) = \lim\limits_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim\limits_{h \to 0} \frac{(2+h)^2 + 1 - (2^2+1)}{h}$

$= \lim\limits_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{1} - \cancel{5}}{h}$

$= \lim\limits_{h \to 0} \frac{\cancel{h}(4 + h)}{\cancel{h}} = 4 \Rightarrow m = 4$

Eqn of tangent line : $y - y_0 = m(x - x_0)$

Eqn of tangent line to $f(x)$ at $x=2$ $\rightarrow (2,5)$, $x_0 = 2$, $y_0 = 5$ $= \; y - y_0 = m(x - x_0)$

$$\boxed{y - 5 = 4(x - 2)}$$

3.2 The Derivative as a Function

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

= depends on "x"

provided the limit exists.

not a certain point anymore! $\Rightarrow f'(x) =$ a function of "x"

Recall

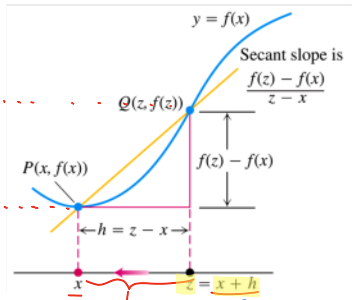
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

particular point

||

slope of tangent line line to $y = f(x)$ at $x = x_0$.

Two forms for the difference quotient.



$z = x + h \rightsquigarrow h = z - x$

$h \to 0$: $x \longrightarrow x+h = z$, $h \to 0$

Derivative of f at x is $\rightarrow$ $f'(x) = \lim_{h \to 0} \dfrac{f(\overbrace{x+h}^{z}) - f(x)}{h}$

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

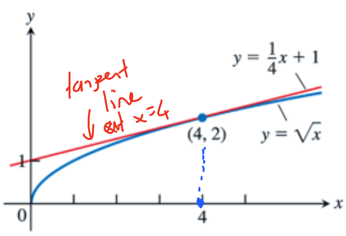
Alternative Formula for the Derivative

$$\boxed{f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.}$$

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.



a) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$ $\sim \frac{0}{0}$

$= \lim_{h \to 0} \dfrac{\cancel{x}+h-\cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

$f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}}$

b) To find the eqn of the tangent line at the point $(4, 2)$ ($x_0 = 4$, $y_0 = 2$):

slope $= m = f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$y - y_0 = m(x - x_0)$

$$y - y_0 = m(x - x_0)$$

$$\boxed{y - 2 = \frac{1}{4}(x - 4)}$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

first derivative fnc. ⇒

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

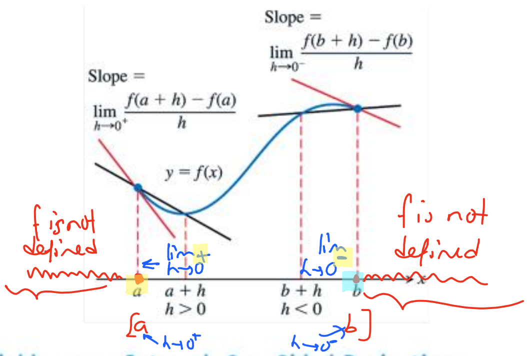
To indicate the value of a derivative at a specified number $x = a$, we use the notation

$x=a$

$$f'(a) = \frac{dy}{dx}\bigg|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{d}{dx}f(x)\bigg|_{x=a}.$$

Figure 3.7

Derivatives at endpoints of a closed interval are one-sided limits.



Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it ha derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ i is differentiable on the interior (a, b) and if the limits

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$f'(b) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7). [redacted]

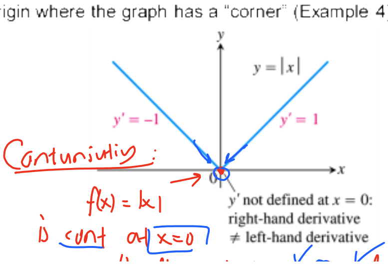
EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

$x=0$ is not included

$f(x) = |x|$ is not differentiable at $x=0$

Figure 3.8

The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).



Continuity:

$f(x) = |x|$ is cont at $\boxed{x=0}$

Continuity at $x=0$

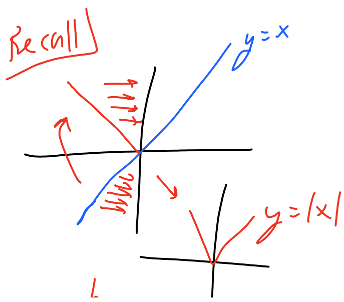
$\lim_{x\to 0} |x| = 0 \overset{?}{=} f(0)$

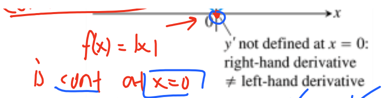
$f(x) = |x|$ is cont at $x=0$

differentiability: at $x=0$

$|h|$

$|0| = 0$

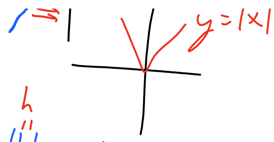




$f(x) = |x|$ is cont at $\boxed{x=0}$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} |x| = 0 = f(0)$

$y(x) = |x|$ is cont at $x = 0$



differentiability: at $x=0$

right-hand derivative at $x=0$: $\lim\limits_{h\to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{|h|}{h} = 1$

left-hand derivative at $x=0$: $\lim\limits_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{|h|}{h} = -1$

$\therefore$ $y = |x|$ is not differentiable at $x=0$

Remark) f is cont at $x=a$ $\not\Rightarrow$ f is differentiable at $x=a$

$\Leftarrow$ ✓

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0$$

$$= \lim_{h\to 0^+} 1 = 1$$

$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h}$$

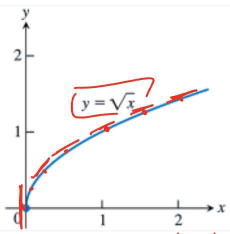
$$= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0$$

$$= \lim_{h\to 0^-} -1 = -1. \qquad \blacksquare$$

Figure 3.9

The square root function is not differentiable at $x = 0$, where the graph of the function has a vertical tangent line.

Ex)



$f(x) = \sqrt{x}$ is not differentiable at $x=0$

has a vertical tangent at $x=0$ ↳ $m = \text{slope} = \infty = f'(0)$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ ⇝ f' is not defined at $x=0$

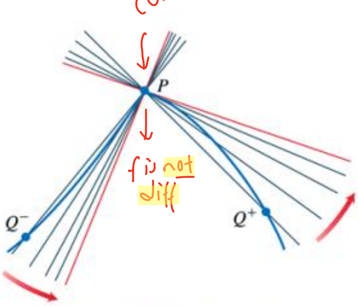
↳ $x=0$ ⇝ $f'(0) = \infty$

Recall

horizontal → $m = 0$

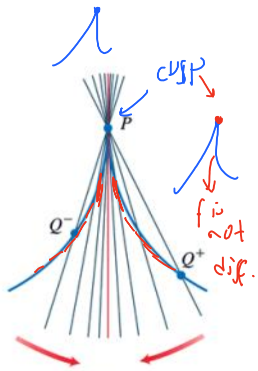
vertical → $m = \infty$

When Does a Function *Not* Have a Derivative at a Point?



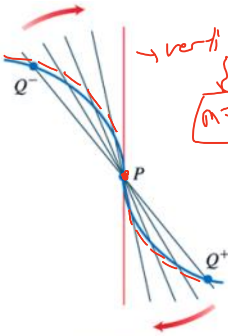
corner

f is not diff



cusp

f is not dif.



→ vertical tangent

$\boxed{m = \infty}$

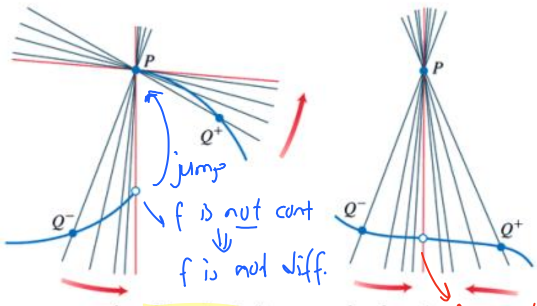
f' is not defined

f is not diff

1. a *corner*, where the one-sided derivatives differ

2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other

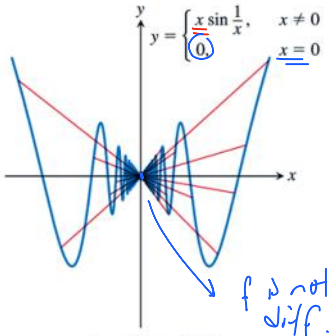
3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



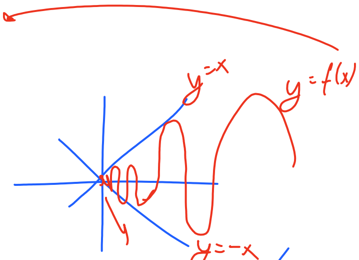
4. a *discontinuity* (two examples shown) f is not cont

⇓

f is not diff.



5. wild oscillation



$\lim_{x \to 0} f(x) = 0 = f(0) \Rightarrow f(x)$ is cont at $x=0$

↑ by Sandwich Thm

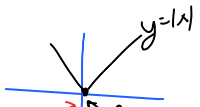
Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1—Differentiability Implies Continuity** If f has a derivative at $x = c$, then f is continuous at $x = c$.

Remark) f is diff $\Longrightarrow$ f is cont

$\not\Longleftarrow$ → Ex) $f(x) = |x|$



corner → f' does not exist

$f(x) = |x| \rightarrow$ cont at $x=0$

but not diff $x=0$

3.3 Differentiation Rules

Derivative of a Constant Function

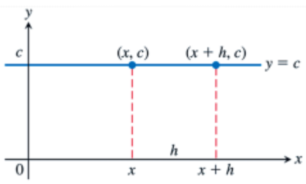
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof) $f'(x) = \lim_{h \to 0} \frac{\overset{c}{f(x+h)} - \overset{c}{f(x)}}{h}$, $f(x) = c$

$= \lim_{h \to 0} \frac{0}{h} = 0$

The rule $\left(\frac{d}{dx}\right)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.



> **Power Rule (General Version)** $f(x) = x^n \leftarrow$ power fnc.
>
> If n is any real number, then
>
> $$\boxed{\frac{d}{dx}x^n = nx^{n-1},} \quad n \in \mathbb{R}$$
>
> for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

a) $y' = 3x^2$

b) $y' = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$

c) $y' = \sqrt{2}x^{\sqrt{2}-1}$

d) $y = x^{-4}$, $y' = -4x^{-5}$

e) $y' = -\frac{4}{3}x^{-\frac{7}{3}}$

f) $y = x^{\frac{2+\pi}{2}} = x^{1+\frac{\pi}{2}}$ → $y' = \left(1+\frac{\pi}{2}\right)x^{\pi/2}$

Derivative Constant Multiple Rule

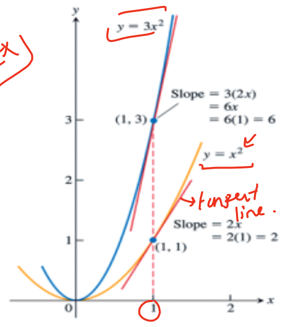
If u is a differentiable function of x, and c is a constant, then

$u(x)$ = a fnc of "x"

$\frac{du}{dx}$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Ex)



The graphs of $y = x^2$ and $y = 3x^2$.

$f(x)$ $g(x)$

slope of the tangent line at $\boxed{x=1}$ $= f'(1) = 2x\big|_{x=1} = 2$

slope of tangent line at $\boxed{x=1}$ $= g'(1) = 6.1 = 6$

$g'(x) = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 6x$

$= 3.2x$

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$u(x)$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Remark)

$(x^1)' = 1.x^{1-1} = 1.x^0 = 1$

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) = \frac{d}{dx}(x^3) + \frac{4}{3}\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x^1) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3}.2x - 5.1 + 0$$

Recall)

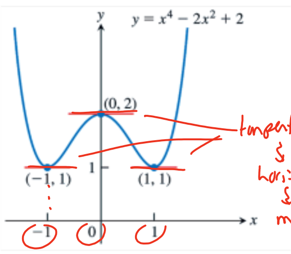
EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

slope = 0

horizontal → slope = 0

vertical line → slope = $\pm\infty$

The curve in Example 4 and its horizontal tangents.



tangent line ⇕ horizontal ⇕ m=0

slope = 0 ⇕ $y' = 4x^3 - 4x = 0$

$4x(x^2-1) = 0$

$4x(x-1)(x+1) = 0$ — $x=0$, $x=1$, $x=-1$ — m=0 ⇓ tangent line is horizontal

Derivative of the Natural Exponential Function

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

exponential fnc.

EXAMPLE 5 Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

$y' = e^x$

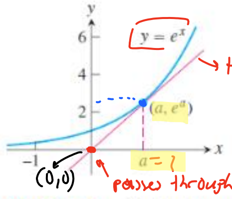


FIGURE 3.13 The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

tangent line at $x = a$ ⇝ slope of tangent line at $\boxed{x = a}$ $= y'(a) = e^a$

passes through origin

• Eqn of this tangent line : $(\overset{x_0}{a}, \overset{y_0}{e^a})$, $m = e^a$

$y - y_0 = m(x - x_0)$

$\boxed{y - e^a = e^a(x - a)}$

tangent line passes through (0,0) : $0 - e^a = e^a(0 - a) \Rightarrow -e^a = -ae^a$ $\boxed{a = 1}$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(\underset{\text{product}}{uv}) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Remark)

$\cancel{(f \cdot g)' = f' \cdot g'}$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

$\boxed{(f \cdot g)' = f' \cdot g + f g'}$

EXAMPLE 6 Find the derivative of **(a)** $y = \underbrace{\frac{1}{x}}_{f(x)} \cdot \underbrace{(x^2 + e^x)}_{g(x)}$,

$y' = f' g + f \cdot g'$ (product rule)

$= -\frac{1}{x^2} \cdot (x^2 + e^x) + \left(\frac{1}{x}\right)(2x + e^x)$

• $\left(\frac{1}{x}\right)' = (x^{-1})' = -1 \cdot x^{-2} = -\frac{1}{x^2}$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$u(x)$
$v(x)$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Remark)

$\cancel{\left(\frac{f}{g}\right)' = \frac{f'}{g'}}$ wrong Quotient Rule

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}}$$

EXAMPLE 8 Find the derivative of **(a)** $y = \frac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

a) $\frac{dy}{dt} = \frac{f'g - fg'}{g^2}$

$= \frac{(2t)(t^3 + 1) - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2}$

b) $y = e^{-t} = \frac{1}{e^t}$ ⇝ $y' = \frac{f'g - fg'}{g^2} = \frac{0 \cdot e^t - 1 \cdot e^t}{(e^t)^2} = -e^{-t}$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $\boxed{f'' = (f')'}$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''=(f')' \rightarrow f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

second derivative

Recall

$y^n \longleftrightarrow y^{(n)}$

n^{th} power n^{th} order derivative

$$y''' = dy''/dx = d^3y/dx^3,$$

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

derivate: $y = f(x)$
$y' = f'(x)$
derivate: $y'' = f''(x)$
derivat: $y''' = f'''(x)$
$\vdots$
n^{th} order derivative $\rightarrow y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x + 0$

Second derivative: $y'' = 6x - 6\cdot 1$

Third derivative: $y''' = 6$

Fourth derivative: $\boxed{y^{(4)} = 0.}$

$\vdots$

37. $y = \sqrt[3]{x^2} - x^e$ **23.** $f(s) = \dfrac{\sqrt{s}-1}{\sqrt{s}+1}$

34. $y = x^{-3/5} + \pi^{3/2}$ **48.** $u = \dfrac{\overbrace{(x^2+x)(x^2-x+1)}^{f}}{\underbrace{x^4}_{g}}$

(48) $u' = \dfrac{f'g - fg'}{g^2} = \dfrac{\overbrace{(2x+1)(x^2-x+1) + (x^2+x)\cdot(2x-1)}^{f'}\cdot \overbrace{x^4}^{g} - \overbrace{(x^2+x)(x^2-x+1)}^{f}\cdot \overbrace{4x^3}^{g'}}{\underbrace{(x^4)^2}_{g^2}}$

3.5 Derivatives of Trigonometric Functions

6 basic trigonometric fncs
sin x
cos x
tan x
cot x
sec x
cosec x

The derivative of the sine function is the cosine function:

$$\boxed{\frac{d}{dx}(\sin x) = \cos x.}$$

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$ Derivative definition

$$= \lim_{h\to 0}\frac{(\sin x\cos h + \cos x \sin h) - \sin x}{h} = \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0}\frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$: $\quad \dfrac{dy}{dx} = 2x - \dfrac{d}{dx}(\sin x) = 2x - \cos x$ — Difference Rule

derivate

$y' = 2x - \cos x$

(b) $y = e^x \sin x$: $\quad \dfrac{dy}{dx} = e^x \dfrac{d}{dx}(\sin x) + \dfrac{d}{dx}(e^x)\sin x = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$ — Product Rule

product

$y' = e^x \cdot \sin x + e^x \cdot (\cos x)$

(c) $y = \dfrac{\sin x}{x}$: $\quad \dfrac{dy}{dx} = \dfrac{x \cdot \dfrac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$ — Quotient Rule ■

f, g

$y' = \dfrac{f'g - fg'}{g^2}$

$= \dfrac{\cos x \cdot x - \sin x \cdot 1}{x^2}$

The derivative of the cosine function is the negative of the sine function:

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x.}$$

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5e^x + \cos x$: $\Rightarrow y' = 5e^x - \sin x$

$\frac{d}{dx}(e^x) = e^x$

(b) $y = \sin x \cos x$: $\Rightarrow y' = \underbrace{\cos x}_{f'} \cdot \underbrace{\cos x}_{g} + \underbrace{\sin x}_{f} \cdot \underbrace{(-\sin x)}_{g'}$

f, g, product

(c) $y = \dfrac{\cos x}{1 - \sin x}$: $\Rightarrow y' = \dfrac{f'g - fg'}{g^2} = \dfrac{-\sin x \cdot (1 - \sin x) - \cos x \cdot (-\cos x)}{(1-\sin x)^2}$

f

Remainder

$\tan x = \dfrac{\sin x}{\cos x}, \quad \cot x = \dfrac{\cos x}{\sin x}, \quad \sec x = \dfrac{1}{\cos x}, \quad$ and $\quad \csc x = \dfrac{1}{\sin x}$

$\cot x = \dfrac{1}{\tan x}$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x \qquad (\sin x)' = \cos x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad (\cos x)' = -\sin x$$

Examples

7. $f(x) = \sin x \tan x$
8. $g(x) = \csc x \cot x$
9. $y = (\underbrace{\sec x + \tan x}_{f})(\underbrace{\sec x - \tan x}_{g})$
10. $y = (\sin x + \cos x)\sec x$
17. $f(x) = x^3 \sin x \cos x$
22. $s = \dfrac{\sin t}{1 - \cos t}$ (f, g)

(7) $f'(x) = \cos x \cdot \tan x + \sin x \cdot \sec^2 x$

(9) $y' = f'g + fg'$

$= (\sec x \tan x + \sec^2 x) \cdot (\sec x - \tan x) + (\sec x + \tan x) \cdot (\sec x \tan x - \sec^2 x)$

(9) $y' = f'g + fg'$

$$= \underbrace{(\sec x \tan x + \sec^2 x)}_{f'} \cdot \underbrace{(\sec x - \tan x)}_{g} + \underbrace{(\sec x + \tan x)}_{f} \cdot \underbrace{(\sec x \tan x - \sec^2 x)}_{g'}$$

(22) $$\frac{ds}{dt} = \frac{f'g - fg'}{g^2} = \frac{\cos t \cdot (1-\cos t) - \sin t \cdot (-(-\sin t))}{(1-\cos t)^2}$$