Problem Solving_sec03

11 Ekim 2020 Pazar   19:14

Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

Recall

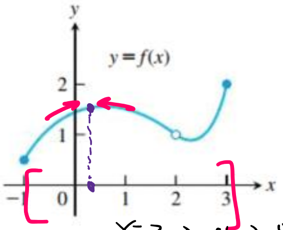
To check continuity at $x=a$

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x) = \text{exist}$ ✓
3. $\lim_{x\to a} f(x) = f(a)$

⇓

$f(x)$ is cont at $x=a$

1.

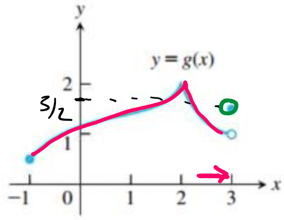


$x=2 \to f(2) = \text{undef}$

⇓

not cont $x=2$

2.



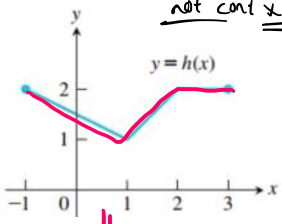
$x=3 \to f(3) = \text{defined}$

$\lim_{x\to 3} f(x) = 1 \neq f(3) = 3/2$

⇓

not cont at $x=3$

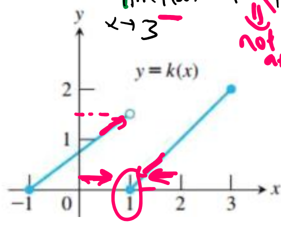
3.



⇓

cont everywhere

4.



$x=1 \to f(1) = 0$

$\lim_{x\to 1} f(x) = \text{DNE}$ (one-sided limits are different)

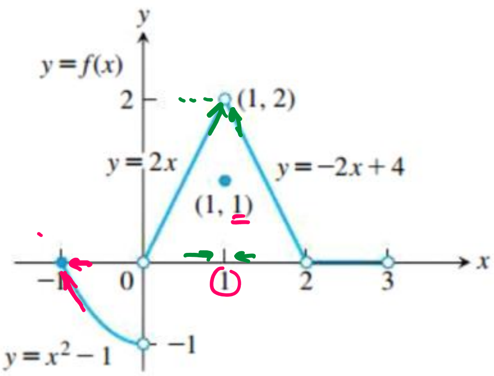
⇓

$f(x)$. not cont at $x=1$

Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. **a.** Does $f(-1)$ exist? Yes $f(-1) = 0$

b. Does $\lim_{x\to -1^+} f(x)$ exist? Yes $\lim_{x\to -1^+} f(x) = 0$

c. Does $\lim_{x\to -1^+} f(x) = f(-1)$? ✓

d. Is f continuous at $x = -1$? Yes.

6. **a.** Does $f(1)$ exist? Yes, $f(1) = 1$

b. Does $\lim_{x\to 1} f(x)$ exist? Yes $\lim_{x\to 1} f(x) = 2$

c. Does $\lim_{x\to 1} f(x) = f(1)$? $2 \neq 1$

d. Is f continuous at $x = 1$? No !

7. **a.** Is f defined at $x = 2$? (Look at the definition of f.) $\to f(2) =$ undef

b. Is f continuous at $x = 2$? $\to$ f not cont at $x=2$

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Limits Involving Trigonometric Functions

Find the limits in Exercises 31–38. Are the functions continuous at the point being approached?

31. $\lim_{x\to\pi} \sin(x - \sin x)$

32. $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$

33. $\lim_{y\to 1} \sec(y\sec^2 y - \tan^2 y - 1)$

34. $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos(\sin x^{1/3})\right)$

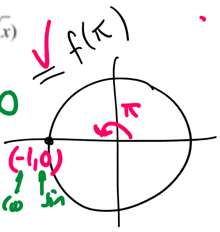
35. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

36. $\lim_{x\to\pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$

37. $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)$

38. $\lim_{x\to 1} \cos^{-1}(\ln\sqrt{x})$

(31) $\lim_{x\to\pi} \underbrace{\sin(x-\sin x)}_{f(x)} = \sin(\pi - \underbrace{\sin\pi}_{0}) = 0 = f(\pi)$



$\Downarrow$

$f(x)$ is cont at $x = \pi$

Past Exam Question

6. (16 points) Let

$$f(x) = \begin{cases} x - c & \text{if } x < 2 \\ b & \text{if } x = 2 \\ -cx^2 + 8 & \text{if } x > 2 \end{cases}$$

(a) (6 points) Find the values of b and c that make the function $f(x)$ continuous at $x = 2$.

continuity at $x=2$:

$f(x) = x-2$ $\quad$ $f(x) = -cx^2+8$ $\quad$ 2

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$

$$\lim_{x\to 2^+} f(x) = -cx^2+8, \quad \lim_{x\to 2^-} f(x) = x-2$$

$$-4c+8 = 2-c \Rightarrow 3c = 6 \Rightarrow \boxed{c=2}$$

$$\lim_{x\to 2} f(x) = 0 = f(2) = b$$

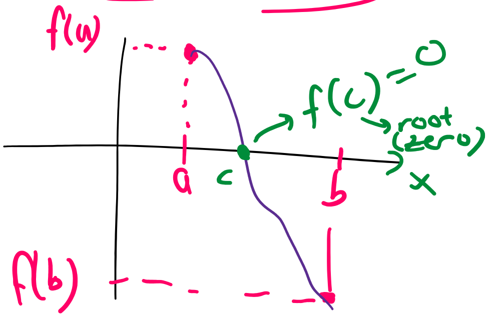
$$\boxed{b=0}$$

Use the Intermediate Value Theorem in Exercises 71–78 to prove that each equation has a solution.

78. $2\sin x = x$ (three roots).

Recall IVT

assumption: $f(x)$ cont on $[a,b]$, $f(a) > 0$, $f(b) < 0$ $\xRightarrow{\text{IVT}}$ conclusion: there is at least one number $c \in (a,b)$ s.t. $f(c) = 0$ ← roots



$f(x) = 2\sin x - x = 0 \rightarrow$ cont everywhere.

$f(0) = 2\sin 0 - 0 = 0 \Rightarrow c_2 = 0$

$f\left(\frac{\pi}{2}\right) = 2\sin\frac{\pi}{2} - \frac{\pi}{2} = 2 - \frac{\pi}{2} > 0$

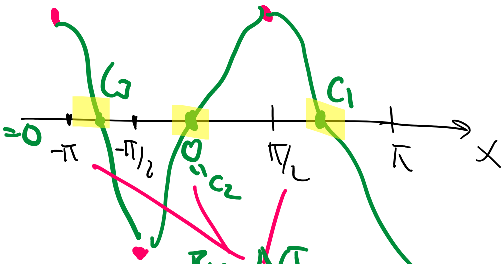
$f(\pi) = 2\underbrace{\sin\pi}_{0} - \pi = -\pi < 0$

by IVT $\Rightarrow f(c_1) = 0$

$f\left(-\frac{\pi}{2}\right) = 2\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = -2 + \frac{\pi}{2} < 0$

$f(-\pi) = 2\underbrace{\sin(-\pi)}_{0} + \pi = \pi > 0$

$\Rightarrow f(c_3) = 0$



By IVT $f(c_1) = f(c_2) = f(c_3) = 0$

roots of $f(x)$

2.6 Limits involving infiny, Asymtotes of graphs

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$.

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$ **18.** $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$ **20.** $g(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

Recall: $\dfrac{\text{number}}{\infty} = 0$, $\dfrac{\text{number}}{0} = \infty$

(17) $\lim\limits_{x\to\infty} \dfrac{7x^3}{x^3 - 3x^2 + 6x} = \lim\limits_{x\to\infty} \dfrac{7\cancel{x^3}}{\cancel{x^3}\left(1 - \frac{3}{x} + \frac{6}{x^2}\right)} = 7$

$\frac{3}{x} \to 0$, $\frac{6}{x^2} \to 0$

∞, $-\infty$

(19) $\lim \dfrac{10x^5 + x^4 + 31}{x^6} = \lim \dfrac{\cancel{x^5}\left(10 + \frac{1}{x} + \frac{31}{x^5}\right)}{\ldots} = \lim \dfrac{10}{x} = 0$

$\frac{1}{x} \to 0$, $\frac{31}{x^5} \to 0$

(19) $\lim_{x\to\infty} \frac{10x^5+x^4+31}{x^6} = \lim_{x\to\infty} \frac{\cancel{x^5}\left(10+\frac{1}{x}+\frac{31}{x^5}\right)}{\cancel{x^6}\, x} = \lim_{x\to\infty} \frac{10}{x} = 0$

$\frac{\infty}{\infty}$

(20) $\lim_{x\to\infty} \frac{x^3+7x^2-2}{x^2-x+1} = \lim_{x\to\infty} \frac{x\,\cancel{x^2}\left(1+\frac{7}{x}-\frac{2}{x^3}\right)}{\cancel{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}\right)} = \lim_{x\to\infty} x = \infty$

$\frac{\infty}{\infty}$

Limits as $x \to \infty$ or $x \to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

→ **33.** $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x\left(1+\frac{1}{x}\right)}$

Recall

$\sqrt{x^2} = |x|$

$= \lim_{x\to\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{1}{x}\right)} = \frac{\sqrt{1}}{1} = 1$

→ **34.** $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to-\infty} \frac{\overbrace{|x|}^{-x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{1}{x}\right)} = \frac{-\sqrt{1}}{1} = -1$

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 86–92. (*Hint*: Try multiplying and dividing by the conjugate.)

86. $\lim_{x\to\infty} (\sqrt{x+9} - \sqrt{x+4})$

87. $\lim_{x\to\infty} (\sqrt{x^2+25} - \sqrt{x^2-1})$

88. $\lim_{x\to-\infty} (\sqrt{x^2+3} + x)$

89. $\lim_{x\to-\infty} (2x + \sqrt{4x^2+3x-2})$

90. $\lim_{x\to\infty} (\sqrt{9x^2-x} - 3x)$

91. $\lim_{x\to\infty} (\sqrt{x^2+3x} - \sqrt{x^2-2x})$

92. $\lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-x})$

(89) $\lim_{x\to-\infty} (2x+\sqrt{4x^2+3x-2}) \sim -\infty+\infty = \underbrace{\infty-\infty}_{\text{indet. infinite form}}$ (not zero!)

$= \lim_{x\to-\infty} \left(2x+\sqrt{4x^2+3x-2}\right) \cdot \left(\frac{2x-\sqrt{4x^2+3x-2}}{2x-\sqrt{4x^2+3x-2}}\right)$

$\cancel{4x^2} - \cancel{4x^2} - 3x + 2 \qquad x\left(-3+\frac{2}{x}\right)$

$$= \lim_{x\to-\infty} \frac{\cancel{4x^2} - \cancel{4x^2} - 3x + 2}{2x - \sqrt{\underbrace{4x^2+3x-2}_{x^2\left(4+\frac{3}{x}-\frac{2}{x^2}\right)}}} = \lim_{x\to-\infty} \frac{x\left(-3+\frac{2}{x}\right)}{2x - \underbrace{|x|}_{-x}\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x}\left(-3+\frac{2}{x}\right)}{\cancel{x}\left(2+\sqrt{4+\frac{3}{x}-\frac{2}{x^2}}\right)} \qquad \frac{2}{x}\to 0,\ \frac{3}{x}\to 0,\ \frac{2}{x^2}\to 0$$

$$= \frac{-3}{2+\sqrt{4}} = -\frac{3}{4} \;//$$

RECALL: Infinite Limits

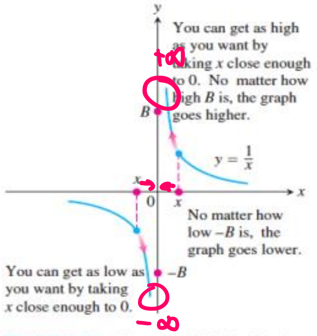


FIGURE 2.59 One-sided infinite limits: $\lim_{x\to0^+} \frac{1}{x} = \infty$ and $\lim_{x\to0^-} \frac{1}{x} = -\infty$.

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.59). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x} = \infty.$$ (still limit does not exist)

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$.*

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53. $\lim \dfrac{1}{x^2-4}$ as

a. $x \to 2^+$ **b.** $x \to 2^-$

c. $x \to -2^+$ **d.** $x \to -2^-$

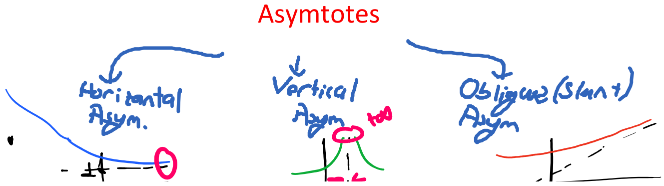
a) $\lim_{x\to2^+} \frac{1}{(x-2)(x+2)} = +\infty$ $\quad (x-2)$: $+0.0001$

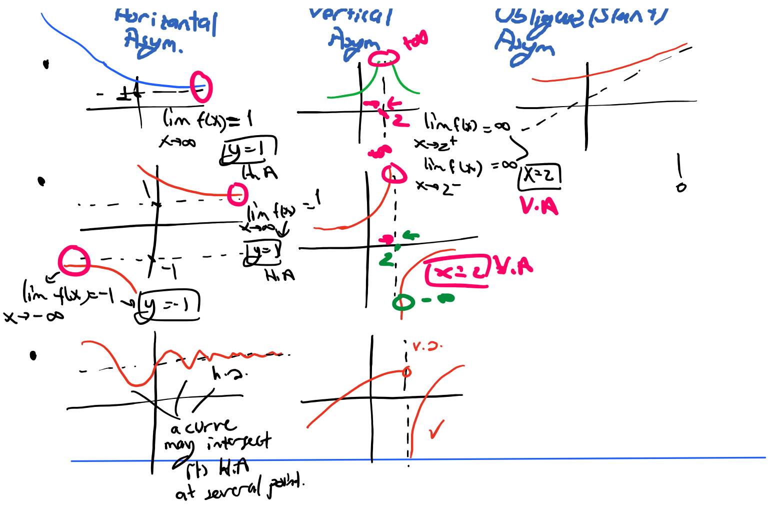
2 ← 2.00001

$\lim_{x\to2^-} \frac{1}{(x-2)(x+2)} = -\infty$ $\quad (x-2)$: -0.00001

1.9999 → 2

Asymptotes

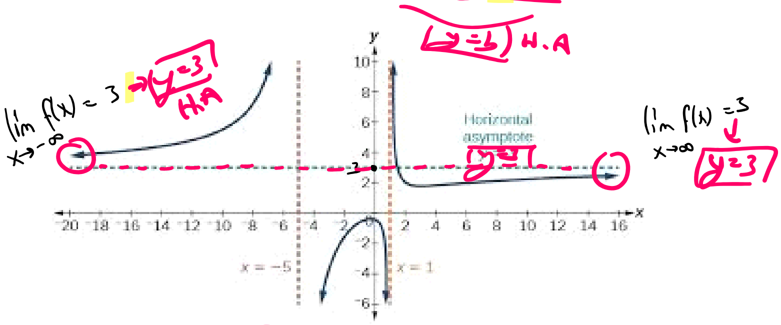




DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

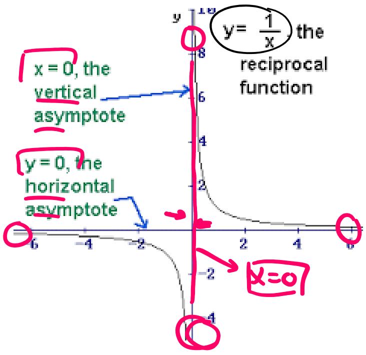
$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

y=b H.A



DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$



$\lim_{x\to 0^+} f(x) = \infty$

$\lim_{x\to 0^-} f(x) = -\infty$

$\Rightarrow$ x=0 V.A

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

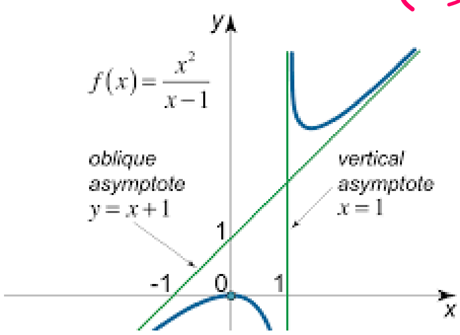
y=0 H.A

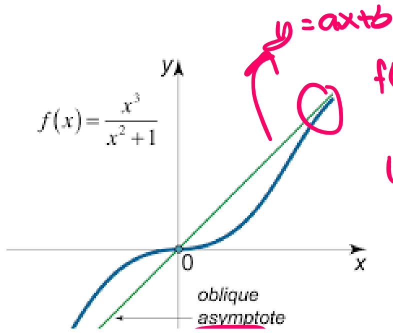
The straight line $y = ax + b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if

$$\text{either} \quad \lim_{x\to-\infty} \big(f(x) - (ax+b)\big) = 0 \quad \text{or} \quad \lim_{x\to\infty} \big(f(x) - (ax+b)\big) = 0,$$

y=ax+b (O.A)

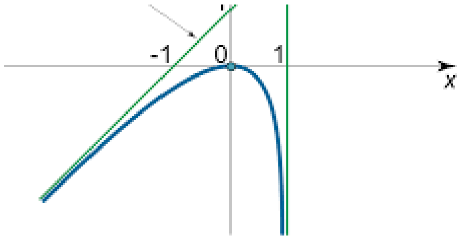
or both.

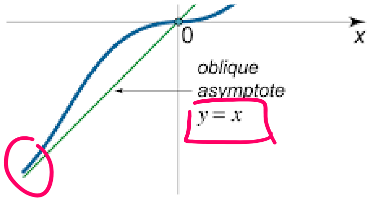




f(x) behaves like the like y=x around x→∞

O.A





O.A

f(x) and $\boxed{y=x}$ are so close $\Big\} \Rightarrow \boxed{y=x}$ O.A

Remarks $f(x) = \dfrac{P(x)}{Q(x)} \rightsquigarrow \boxed{\deg P(x) = \deg Q(x) + 1}$

$\Downarrow$

there is an oblique asym.

$\Downarrow$

How to find O.A ?

$$\begin{array}{r|l} P(x) & Q(x) \\ \vdots & \overline{\underbrace{ax+b}} \\ - \quad \overline{\;} & \\ R(x) & \end{array}$$

$\Downarrow$ $y = ax+b$ O.A

$$\frac{P(x)}{Q(x)} = \underbrace{ax+b}_{\text{O.A.}} + \frac{R(x)}{Q(x)}$$

Ex) $f(x) = \dfrac{x^2}{x-1}$ $\begin{matrix}\to \deg = 2 \\ \to \deg = 1\end{matrix} \Big\}$ so there is an O.A

• O.A

$$\begin{array}{r|l} x^2 & x-1 \\ -\; x^2 - x & \overline{\underbrace{x+1}} \\ x & \\ -\; x-1 & \\ \hline +1 & \end{array}$$

$\Downarrow$ $y = x+1$ O.A

$$f(x) = \frac{x^2}{x-1} = \underbrace{x+1}_{\text{O.A.}} + \frac{1}{x-1}$$

• V.A $f(x) = \dfrac{x^2}{\underbrace{x-1}}$

→ search for the zeros of the denom.: $x-1=0$

$\boxed{x=1}$

candidate for V.A:

$$\left.\begin{array}{l} \lim\limits_{x \to 1^+} \dfrac{x^2}{x-1} = \infty \\ \lim\limits_{x \to 1^-} \dfrac{x^2}{x-1} = -\infty \end{array}\right\} \boxed{x=1} \text{ V.A}$$

EXAMPLE 15 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$ → • no O.A

• H.A

$$\lim_{x\to\infty} \frac{x+3}{x+2} = \lim_{x\to\infty} \frac{\cancel{x}(1+\frac{3}{x})}{\cancel{x}(1+\frac{2}{x})} = 1 \Big) \boxed{y=1} \text{ H.A}$$

$$\lim \frac{x+3}{x+2} = \ldots\ldots 1$$

y

$$\lim_{x\to -\infty} \frac{x+3}{x+2} = \dots = 1$$

$\boxed{y=1}$ H.A

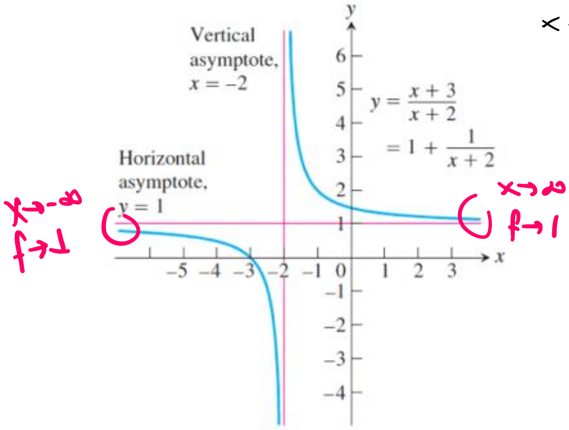


FIGURE 2.65 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 15.

• VA : $y = \frac{x+3}{x+2}$

search for the zero's of den:

$x+2=0 \Rightarrow \boxed{x=-2}$ candidate

$$\lim_{x\to -2^+} \frac{x+3}{x+2} = \infty$$

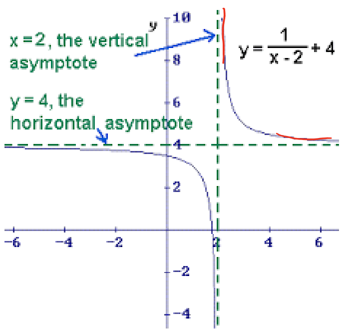
$-2 \leftarrow -1.9999$

$$\lim_{x\to -2^-} \frac{x+3}{x+2} = -\infty$$

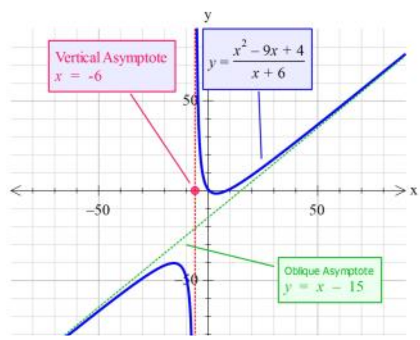
$-2.00011 \rightarrow -2$

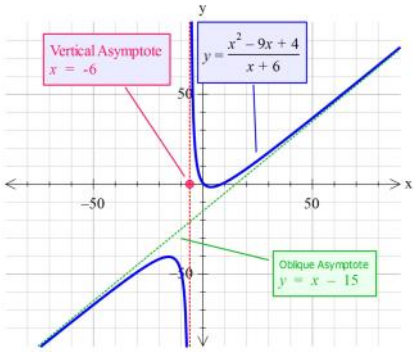
$\boxed{x=-2}$ V.A

Ex)

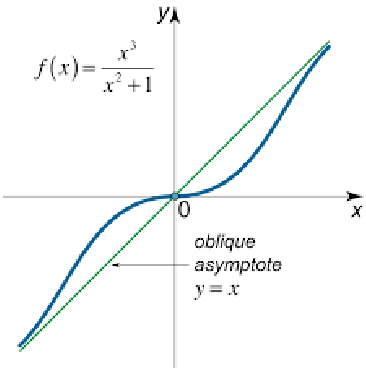


Ex)

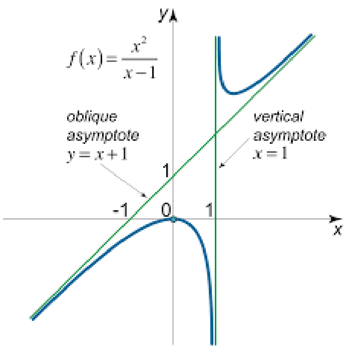



y
Vertical Asymptote
x = -6
$y = \frac{x^2 - 9x + 4}{x + 6}$
50
-50
50
x
Oblique Asymptote
y = x - 15

Ex)


y
$f(x) = \frac{x^3}{x^2+1}$
0
x
oblique
asymptote
$y = x$

Ex)


y
$f(x) = \frac{x^2}{x-1}$
oblique
asymptote
$y = x + 1$
vertical
asymptote
$x = 1$
1
-1
0
1
x

Ex)

Find the oblique asymptote of graphs of the rational functions

99. $y = \dfrac{x^2}{x - 1}$

100. $y = \dfrac{x^2 + 1}{x - 1}$

101. $y = \dfrac{x^2 - 4}{x - 1}$

102. $y = \dfrac{x^2 - 1}{2x + 4}$

103. $y = \dfrac{x^2 - 1}{x}$

104. $y = \dfrac{x^3 + 1}{x^2}$

Past Exam Question

4. (15 points) Find all asymptotes of the graph $y = \dfrac{(x+3)(x+2)(x+1)}{x^2 + x - 2}$, if exists. Classify them as vertical, horizontal and oblique (i.e. slant).