Continuity

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

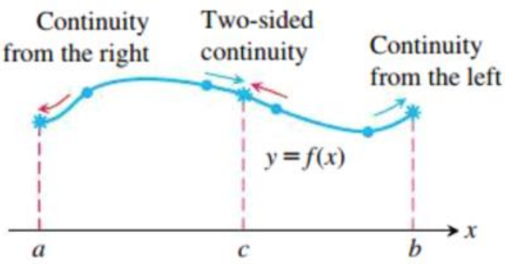


FIGURE 2.57 Continuity at points a, b, and c.

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f. Note that c need not be in the domain of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36).

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

EXAMPLE 1 At which numbers does the function f in Figure 2.56 appear to be continuous? Explain why. What occurs at other numbers in the domain?

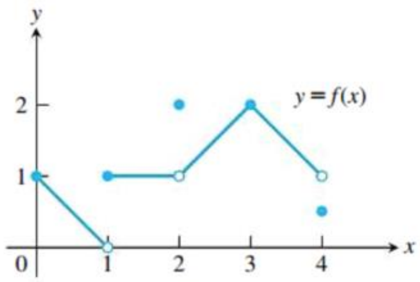


FIGURE 2.56 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.59, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

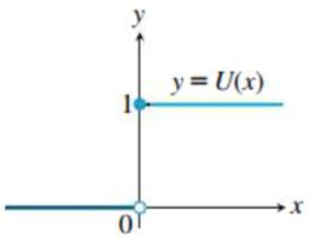


FIGURE 2.59 A function that has a jump discontinuity at the origin (Example 3).

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. For example, the semicircle function graphed in Figure 2.37 is continuous on the interval $[-2, 2]$, which is its domain. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.58). It is right-continuous at $x = -2$, and left-continuous at $x = 2$. ■

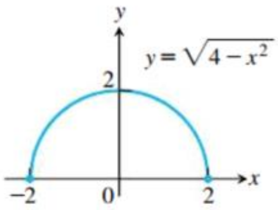


FIGURE 2.58 A function that is continuous over its domain (Example 2).

EXAMPLE 5

(a) The function $y = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. It has a point of discontinuity at $x = 0$, however, because it is not defined there; that is, it is discontinuous on any interval containing $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

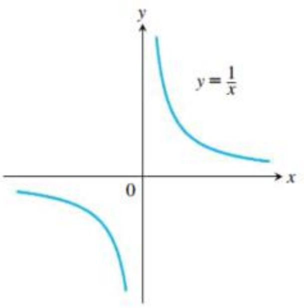


FIGURE 2.62 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

Ex)

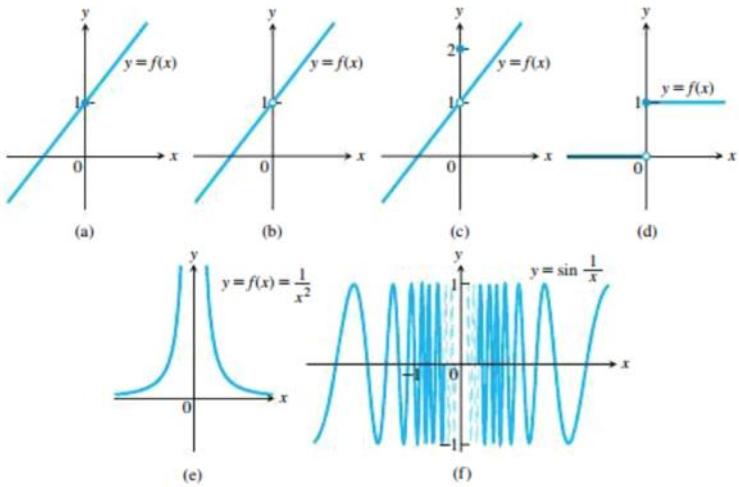


FIGURE 2.61 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Properties of Continuous Functions

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Composite of Continuous Functions

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

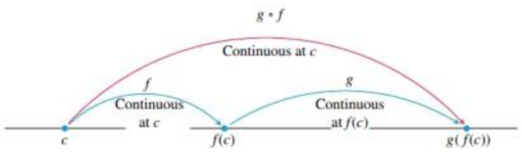


FIGURE 2.63 Compositions of continuous functions are continuous.

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

EXAMPLE 9 As an application of Theorem 11, we have the following calculations.

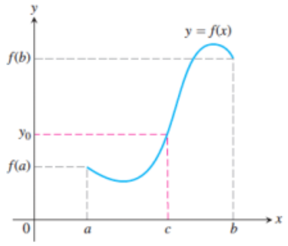
(a)
$$\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x \to \pi/2} 2x + \lim_{x \to \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right)$$
$$= \cos(\pi + \sin 2\pi) = \cos \pi = -1.$$

(b)
$$\lim_{x \to 1} \sin^{-1}\left(\frac{1 - x}{1 - x^2}\right) = \sin^{-1}\left(\lim_{x \to 1} \frac{1 - x}{1 - x^2}\right)$$ Arcsine is continuous.
$$= \sin^{-1}\left(\lim_{x \to 1} \frac{1}{1 + x}\right)$$ Cancel common factor $(1 - x)$.
$$= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

(c)
$$\lim_{x \to 0} \sqrt{x + 1}\, e^{\tan x} = \lim_{x \to 0} \sqrt{x + 1} \cdot \exp\left(\lim_{x \to 0} \tan x\right)$$ exp is continuous.
$$= 1 \cdot e^0 = 1$$ ■

Intermediate Value Theorem

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps like the graph of the greatest integer function (Figure 2.39), or separate branches like the graph of $1/x$ (Figure 2.41).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

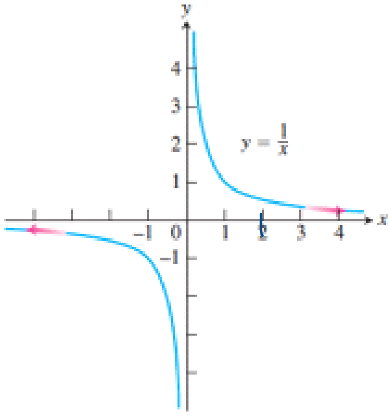
55. Roots of a cubic Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

Limits Involving Infinity; Asymptotes of Graphs

- The symbol for infinity ∞ does not represent a real number.
- We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$



THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c.

(a) $\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x}$ Sum Rule

$= 5 + 0 = 5$ Known limits

(b) $\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$

$= \lim_{x\to-\infty} \pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}$ Product Rule

$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$ Known limits ■

Strategy: Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator.

31. $$\lim_{x\to\infty} \frac{2x^{5/3} + x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$$

32. $$\lim_{x\to-\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$$

Horizontal Asymptotes

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLE 5 The x-axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x \to -\infty} e^x = 0.$$

EXAMPLE 6 Find **(a)** $\lim_{x \to \infty} \sin(1/x)$ and **(b)** $\lim_{x \to \pm\infty} x \sin(1/x)$.

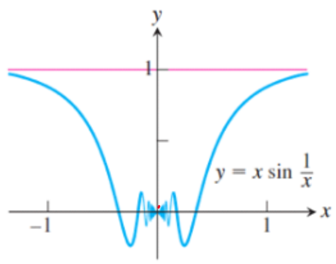
(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.49). Therefore,

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} x \sin \frac{1}{x} = \lim_{t \to 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.55, and we see that the line $y = 1$ is a horizontal asymptote. ■



★ The Sandwich Theorem also holds for limits as $x \to \pm\infty$.

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty} |1/x| = 0$, we have $\lim_{x\to\pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.57).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

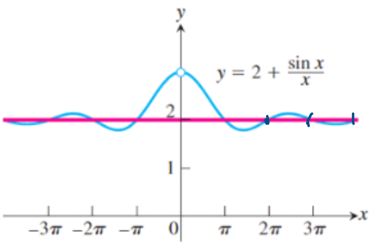


FIGURE 2.57 A curve may cross one of its asymptotes infinitely often (Example 8).

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

The straight line $y = ax + b$ (where $a \neq 0$) is an **oblique asymptote** of the graph of $y = f(x)$ if

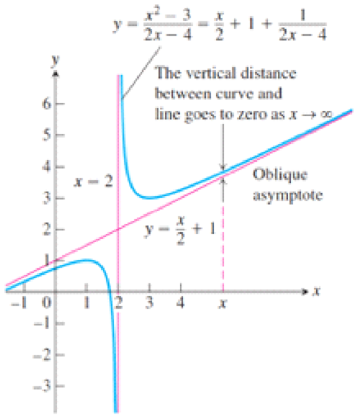
either $\lim_{x\to-\infty}\big(f(x) - (ax+b)\big) = 0$ or $\lim_{x\to\infty}\big(f(x) - (ax+b)\big) = 0,$

or both.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.58.



Infinity Limit

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.59). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

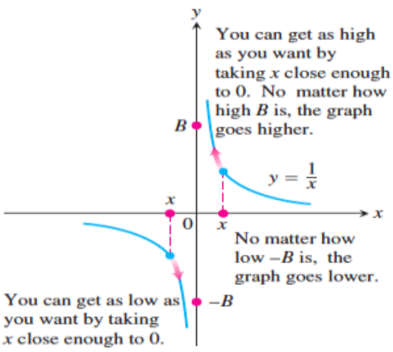
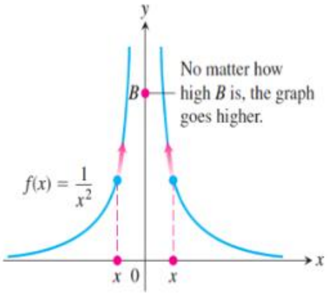
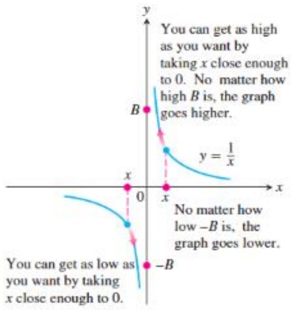


FIGURE 2.59 One-sided infinite limits: $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \to 0.$$





EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} =$

(b) $\lim_{x\to 2} \frac{x-2}{x^2-4} =$

(c) $\lim_{x\to 2^+} \frac{x-3}{x^2-4} =$

(d) $\lim_{x\to 2^-} \frac{x-3}{x^2-4} =$

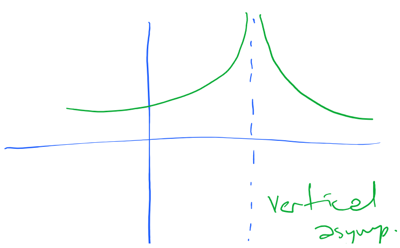
(e) $\lim_{x\to 2} \frac{x-3}{x^2-4} =$

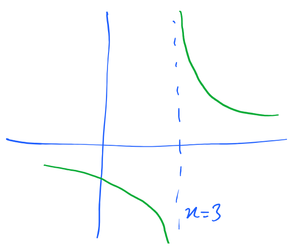
(f) $\lim_{x\to 2} \frac{2-x}{(x-2)^3} =$

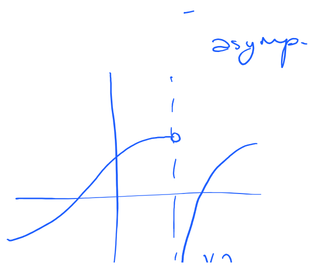
Vertical Asymptotes

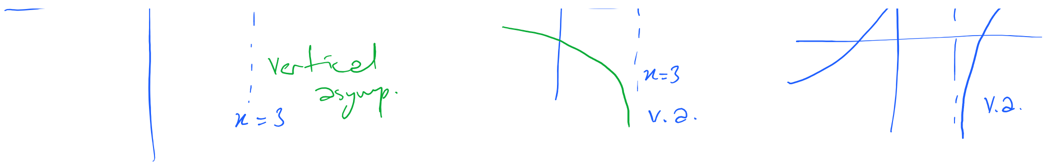
DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty .$$









EXAMPLE 15 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

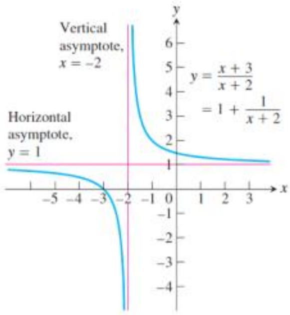


FIGURE 2.65 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 15.

EXAMPLE 17 The graph of the natural logarithm function has the y-axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.67 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x-axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x \to 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

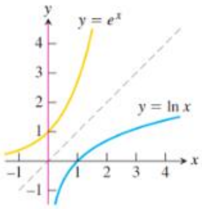


FIGURE 2.67 The line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 17).

EXAMPLE 18 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.68).

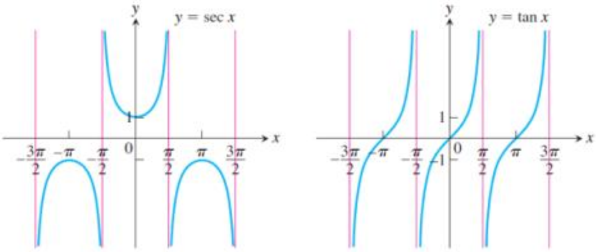
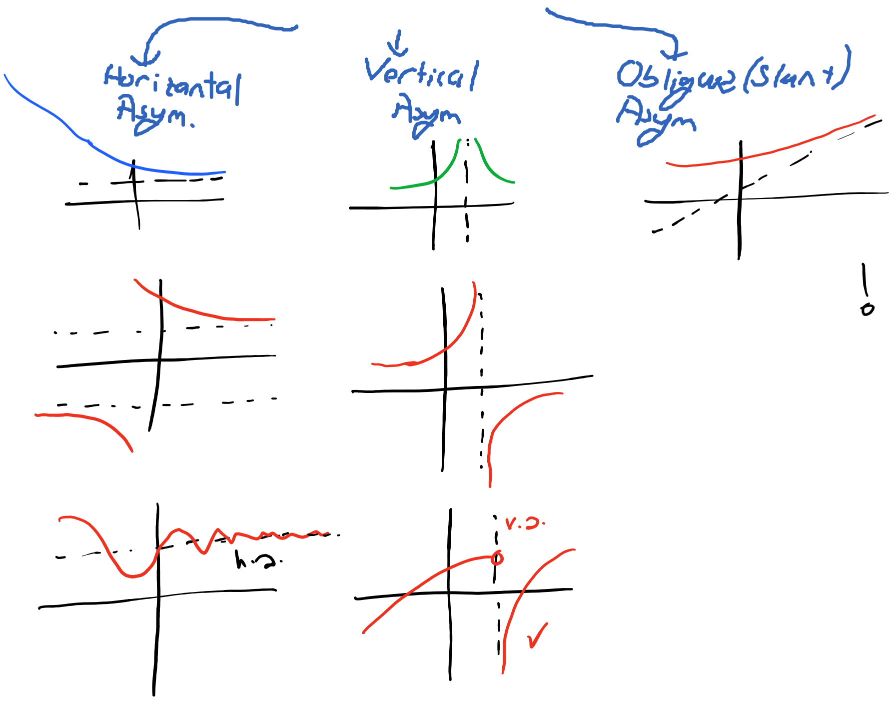
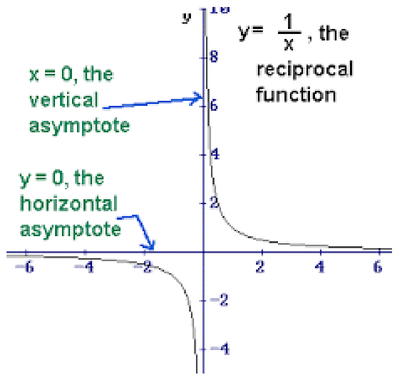
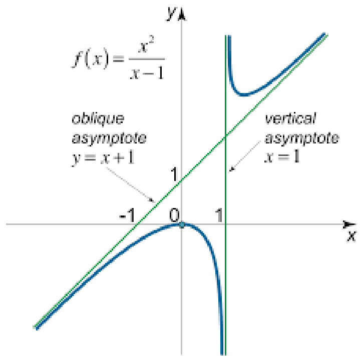


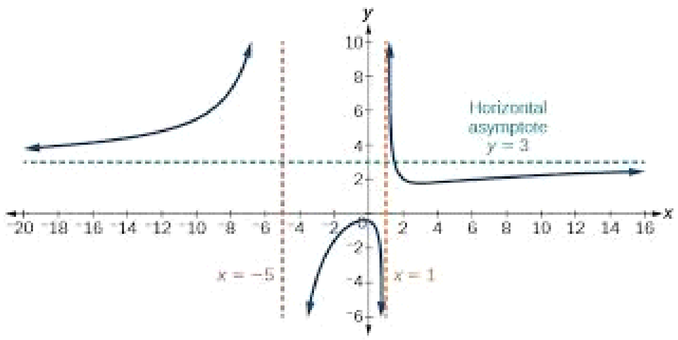
FIGURE 2.68 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 18).

Asymptotes




y = 1/x, the reciprocal function
x = 0, the vertical asymptote
y = 0, the horizontal asymptote


$f(x) = \frac{x^2}{x-1}$
oblique asymptote $y = x + 1$
vertical asymptote $x = 1$


Horizontal asymptote y = 3
x = −5
x = 1