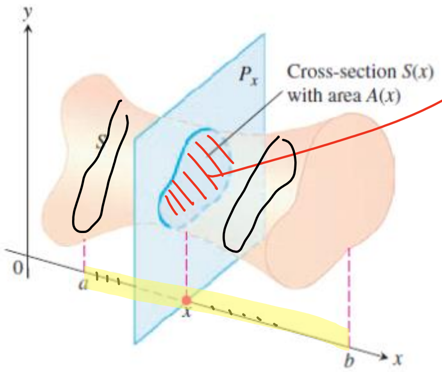
Lecture

Saturday, December 19, 2020 7:13 PM

6 Applications of Definite Integrals

By using definite integral: (1) Area under the curve
(2) Volumes

6.1 Volumes Using Cross-Sections



A(x) = area of the cross section

$$\text{Volume} = \int_a^b A(x)\,dx$$

FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

Slicing by Parallel Planes

zoom into

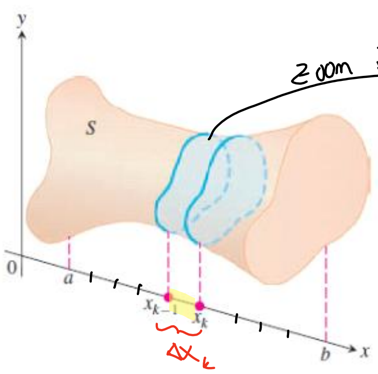


FIGURE 6.3 A typical thin slab in the solid S.

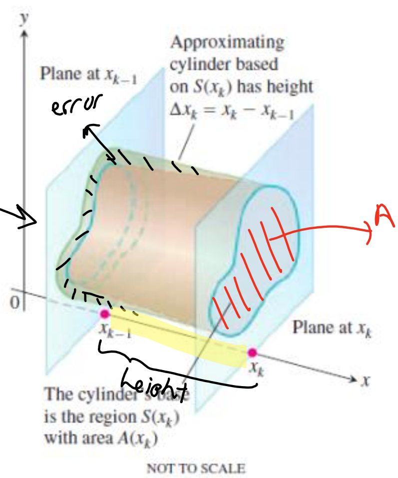


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

$V_k = \underbrace{\text{base}} \times \underbrace{\text{height}}$

$V_k = A(x) \times \Delta x_k$

$V_k = A(x) \cdot (x_k - x_{k-1})$

The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

Volume of solid S

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k .$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$\boxed{V = \int_a^b A(x)\,dx.}$$

Calculating the Volume of a Solid

A(x)

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$*, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate $A(x)$ to* find the volume.

$V = \int_a^b A(x)\,dx$

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

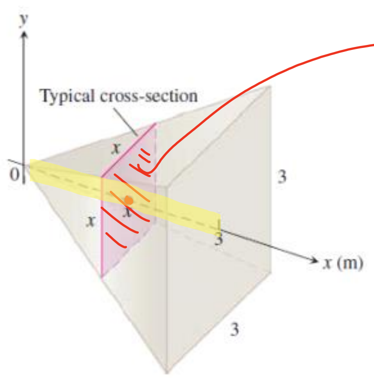


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

$A(x) =$ cross section (square) $= x^2$

$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx$$

$$= \frac{x^3}{3}\Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3}$$

$$= 9\ m^3 \;//$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

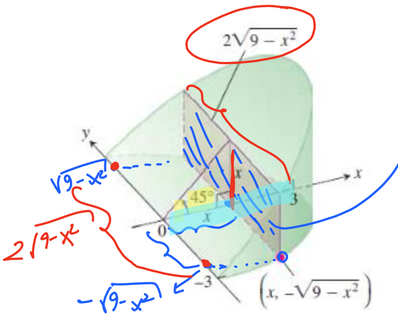


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

$A(x)$ = area of rectangle = base × height

$= 2\sqrt{9-x^2} \cdot x$

$$\text{Volume} = \int_0^3 A(x)\,dx = \int_0^3 2x\sqrt{\underbrace{9-x^2}_{u}}\,dx = -\int_9^0 \sqrt{u}\,du$$

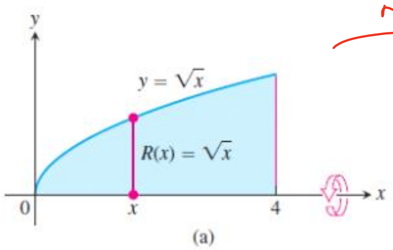
$u = 9 - x^2$, $x=0 \to u=9$, $x=3 \to u=0$

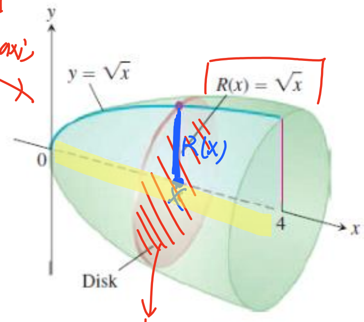
$du = -2x\,dx$

$$= -\frac{u^{3/2}}{3/2}\Big|_9^0 = -0 + \frac{2}{3}9^{3/2}$$

Solids of Revolution: The Disk Method

revolve the shaded region around x-axis





$A(x)$ = area of a disk with $R(x)$

$= \pi(R(x))^2 = \pi(\sqrt{x})^2$

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

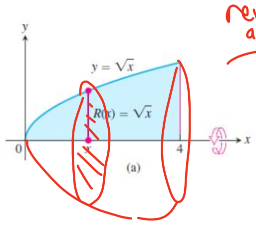
$$\text{Volume} = \int_0^4 A(x)\,dx = \int_0^4 \pi\,(R(x))^2\,dx = \int_0^4 \pi(\sqrt{x})^2\,dx$$

Volume by Disks for Rotation About the x-axis

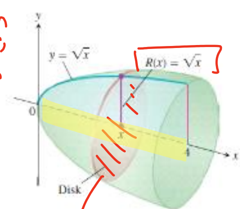
$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

disks with $R(x)$

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



revolve region around x-axis



$A(x) = \pi\,(R(x))^2 = \pi\,(\sqrt{x})^2$

$$\text{Volume} = \int_0^4 A(x)\,dx = \int_0^4 \pi\,(\sqrt{x})^2\,dx$$

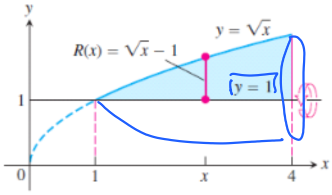
$$= \int_0^4 \pi x\,dx$$

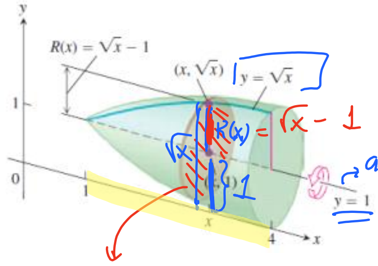
$$= \int_0^4 \pi x \, dx$$

$$= \pi \frac{x^2}{2} \Big|_0^4 = 8\pi$$

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $\boxed{y = 1}$.

revolve the region around $\boxed{y=1}$ line





$$A(x) = \pi (R(x))^2 = \pi (\sqrt{x} - 1)^2$$

$$\text{Volume} = \int_1^4 A(x) \, dx = \int_1^4 \pi (\sqrt{x} - 1)^2 dx$$

$$= \pi \int_1^4 x - 2\sqrt{x} + 1 \, dx \qquad (\sqrt{x} = x^{1/2})$$

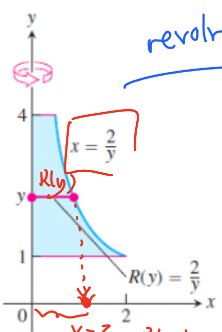
$$= \pi \left(\frac{x^2}{2} - 2 \frac{x^{3/2}}{3/2} + x \right) \Big|_1^4 = \frac{7\pi}{6}$$

Volume by Disks for Rotation About the y-axis

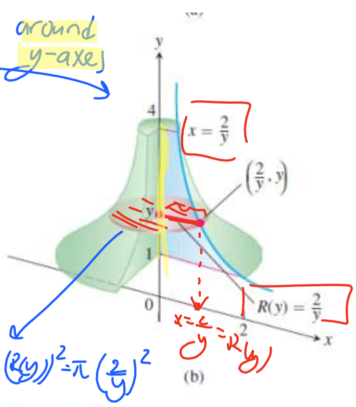
$$V = \int_c^d A(y) \, dy = \int_c^d \pi [R(y)]^2 \, dy.$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

revolve region around y-axes



$x = \frac{2}{y} = R(y)$



$x = \frac{2}{y} = R(y)$

$A(y) = \pi (R(y))^2 = \pi \left(\frac{2}{y}\right)^2$

FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

$$\text{Volume} = \int_1^4 A(y) \, dy = \int_1^4 \pi (R(y))^2 dy$$

$$\text{Volume} = \int_1^4 A(y)\,dy = \int_1^4 \pi\,(R(y))^2\,dy$$

$$= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \pi\int_1^4 \frac{4}{y^2}\,dy$$

$$= \pi\int_1^4 4y^{-2}\,dy = \pi\, 4\,\frac{y^{-1}}{-1}\Big|_{y=1}^{y=4} = 3\pi \,//$$

Solids of Revolution: The Washer Method

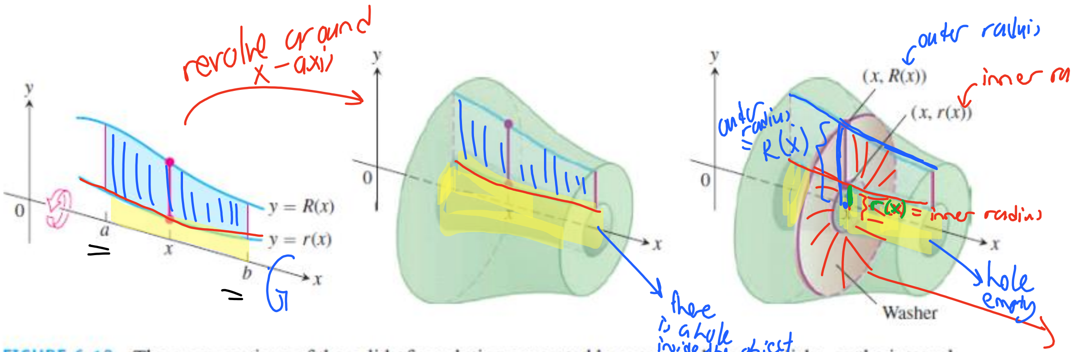
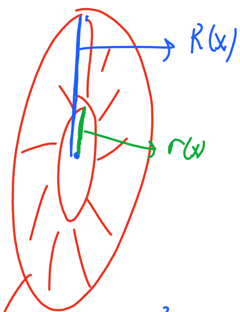


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.



Outer radius: $R(x)$
Inner radius: $r(x)$

The washer's area is

cross section

$$A(x) = \text{Area of ring} = \pi R(x)^2 - \pi r(x)^2 = \pi\left[R(x)^2 - r(x)^2\right]$$

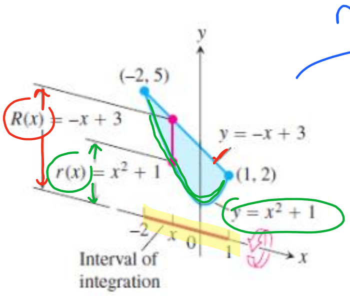
$$V = \int_a^b A(x)\,dx$$

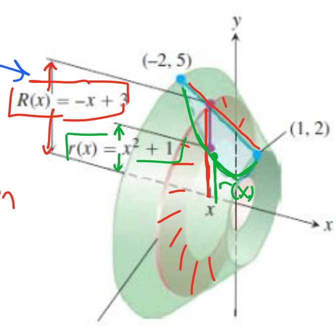
Volume by Washers for Rotation About the x-axis

$$V = \int_a^b \underbrace{A(x)}_{\text{area of the ring}}\,dx = \int_a^b \pi\big(\underbrace{[R(x)]^2}_{\text{outer radius}} - \underbrace{[r(x)]^2}_{\text{inner radius}}\big)\,dx.$$

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

revolve region around x-axis.





Washer cross-section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

cross section

Washer method

$$A(x) = \text{area of ring}$$

$$= \pi R(x)^2 - \pi r(x)^2$$

$$= \pi[\ldots]$$

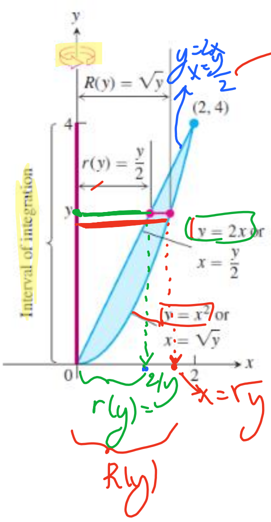
$$= \pi R(x)^2 - \pi r(x)^2$$

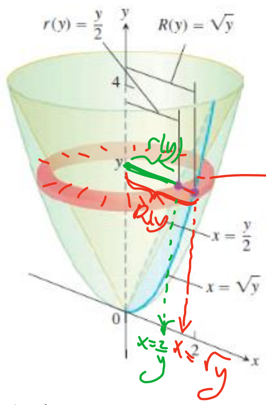
$$= \pi\left[(-x+3)^2 - (x^2+1)^2\right]$$

$$\text{Volume} = \int_{-2}^{1} A(x)\,dx = \int_{-2}^{1} \pi\left[(-x+3)^2 - (x^2+1)^2\right]dx$$

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

revolve the region around y-axis





$A(x) =$ area of this ring

$$= \pi R(y)^2 - \pi r(y)^2$$

$$= \pi\left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right]$$

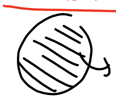
Washer Method

$$V = \int_0^4 A(y)\,dy = \int_0^4 \pi\left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right]dy = \frac{8\pi}{3}$$

6.2 Volumes Using Cylindrical Shells

Volumes obtained by revolving region

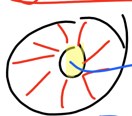
① Disk Method



$A(x) = \pi R^2(x)$

$V = \int_a^b A(x)\,dx$

② Washer Method

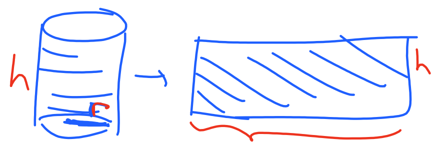


hole

$A(x) = \pi R^2(x) - \pi r^2(x)$

$V = \int_a^b A(x)\,dx$

③ Cylindrical Shell Method



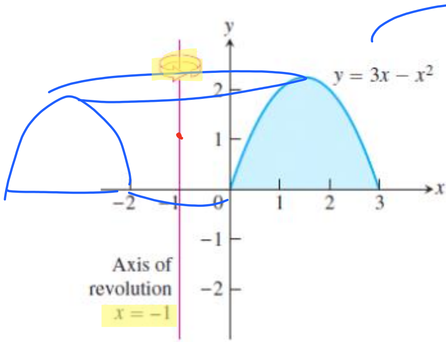
$A(x) = 2\pi r \cdot h$

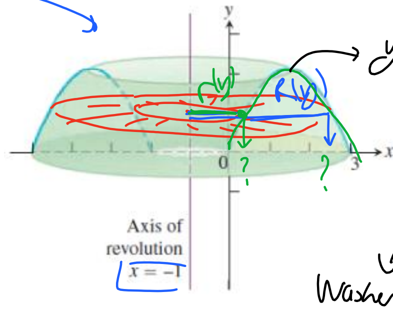
$V = \int_a^b A(x) = \int_a^b 2\pi r h\,dx$

Slicing with Cylinders

EXAMPLE 1 The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (Figure 6.16). Find the volume of the solid.

revolve this region around $x=-1$ line





$y = 3x - x^2$

$x = ?$

We cannot solve x in terms of y

$\Downarrow$

Washer method is not applicable

Use Cylindrical Shell Method:

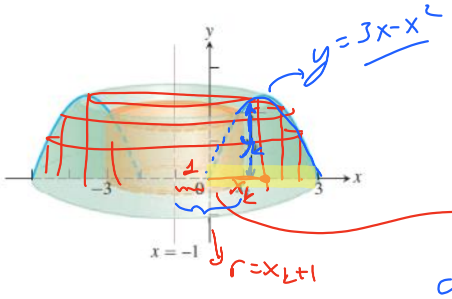
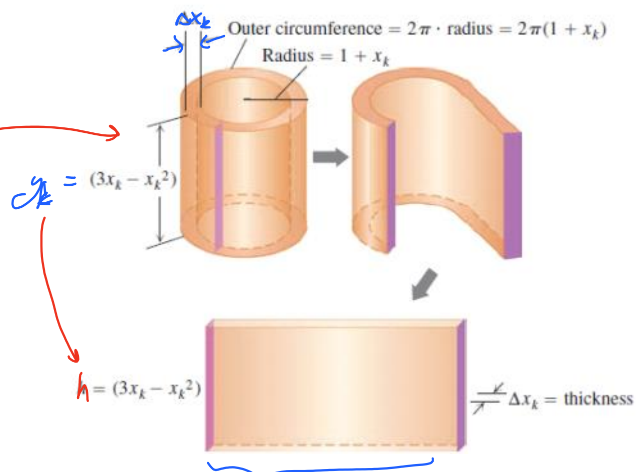


FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).



volume of k^{th} cylindrical shell

$$\Delta V_k = \text{circumference} \times \text{height} \times \text{thickness}$$
$$= 2\pi(1 + x_k) \cdot (3x_k - x_k^2) \cdot \Delta x_k.$$

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)(3x_k - x_k^2)\, \Delta x_k.$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$V \approx \sum_{k=1}^{n} \Delta V_k \xrightarrow{n\to\infty} V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi \underbrace{(x_k + 1)}_{\text{radius}} \underbrace{(3x_k - x_k^2)}_{\text{height}} \underbrace{\Delta x_k}_{\text{thickness}}$$

$$= \int_0^3 2\pi \underbrace{(x + 1)}_{\text{radius}} \underbrace{(3x - x^2)}_{\text{height}}\, dx$$

$$= \int_0^3 2\pi (3x^2 + 3x - x^3 - x^2)\, dx$$

$$= 2\pi \int_0^3 (2x^2 + 3x - x^3)\, dx$$

$$= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4 \right]_0^3 = \frac{45\pi}{2}.$$

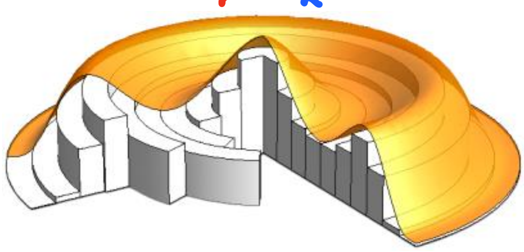
$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \int_a^b 2\pi \underbrace{(\text{shell radius})}_{r} \underbrace{(\text{shell height})}_{h}\, dx.$$

The Shell Method

Shell Formula for Revolution About a Vertical Line

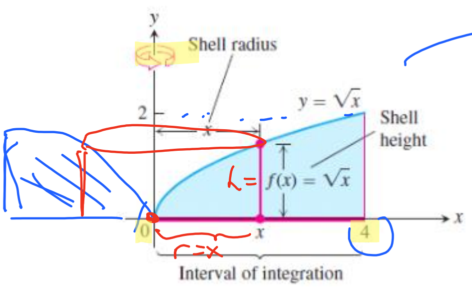
The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \ge 0$, $L \le a \le x \le b$, about a vertical line $x = L$ is

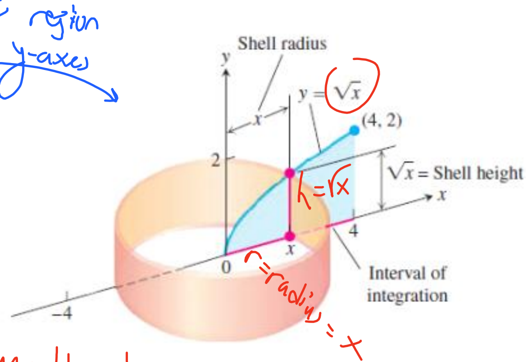
$$V = \int_a^b 2\pi \underbrace{\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}}_{r} \underbrace{\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}}_{h} dx. = \int_a^b 2\pi r h \, dx$$



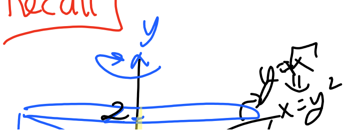
EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

revolve the region around y-axes





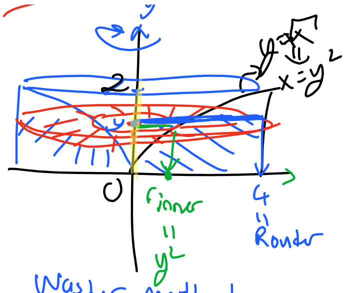
Recall



Shell method

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

4, '' ''



$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$V = \int_0^4 2\pi \; r \; h \; dx$$

$$= \int_0^4 2\pi \; x \; (\sqrt{x}) \; dx$$

$$= 2\pi \int_0^4 x^{3/2} \, dx = 2\pi \frac{x^{5/2}}{5/2}\Big|_0^4 = \frac{128\pi}{5}$$

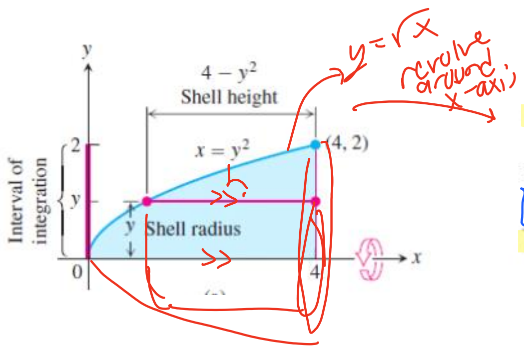
Washer Method

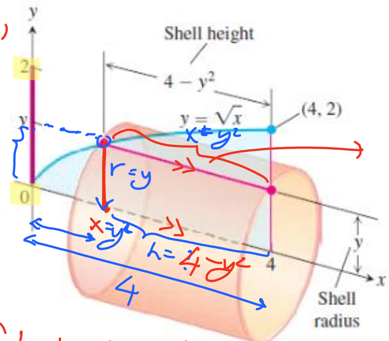
$$V = \int_0^2 A(y)\, dy$$

$$= \int_0^2 \pi \left(R^2(y) - r^2(y) \right) dy$$

$$= \int_0^2 \pi \left(4^2 - (y^2)^2 \right) dy = \frac{128\pi}{5}$$

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.



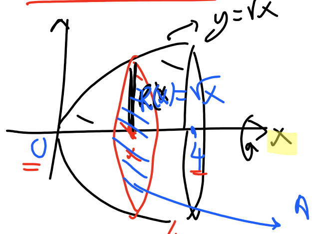


draw a line passing through your region which is parallel to axes-of-revolution

⇓

this line will be the height of your cylinder!

Disk Method



$$A(x) = \pi R^2(x) = \pi(\sqrt{x})^2$$

$$\text{Volume} = \int_0^4 A(x)\, dx$$

$$= \int_0^4 \pi (\sqrt{x})^2 dx$$

$$= \int_0^4 \pi x \, dx$$

$$= \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$$

Cylindrical Shell Method

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy$$

$r = y$, $h = (4 - y^2)$

$$V = \int_0^2 2\pi \cdot y \cdot (4 - y^2)\, dy$$

$$V = 2\pi \int_0^2 4y - y^3 \, dy$$

$$V = 2\pi \left(\frac{4y^2}{2} - \frac{y^4}{4} \right)\Big|_0^2$$

$$= 8\pi$$

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these: h = height

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

$V = \int_a^b 2\pi \, r \, h \, dx$

r = shell's radius, h = shell's height