Lecture

10 Aralık 2020 Perşembe 21:19

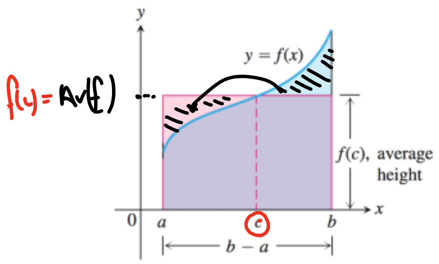
5.4 The Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$\underbrace{f(c)}_{f_{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$f(c)$
$\|$
$$\overline{Av}(f) = \frac{\int_a^b f(x)\,dx}{b-a}$$
$\Downarrow$
$$\int_0^b f(x)\,dx = Av(f)\cdot(b-a)$$

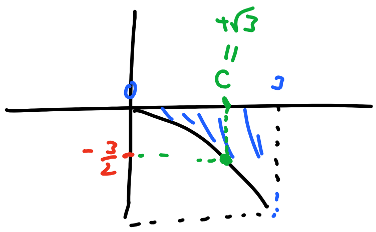


Ex) $f(x) = -\frac{x^2}{2}$ on $[0,3]$

$$Av(f) = \frac{\int_0^3 -\frac{x^2}{2}\,dx}{3-0}$$

$$= \frac{-\frac{1}{2}\,\frac{x^3}{3}\Big|_0^3}{3} = \frac{-\frac{1}{2}\cdot\frac{3^3}{3} + \frac{1}{2}\,\frac{0^3}{3}}{3} = -\frac{3}{2}$$

$$f(c) = -\frac{c^2}{2} = -\frac{3}{2} \Rightarrow c^2 = 3 \Rightarrow c = +\sqrt{3} \text{ or } \cancel{c = -\sqrt{3}}$$



Calculus

Differentiation $\xleftrightarrow[\text{operations}]{\text{inverse}}$ Integration

FTC

FTC I:

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

FTC II

$$\boxed{\int_a^b f'(x) = f(b) - f(a)}$$

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x). \qquad (2)$$

(a) + constant

depends on x

$$\frac{d}{dx}\left(\underbrace{\int_a^x f(t)\,dt}_{F(t)\big|_a^x}\right) = \frac{d}{dx}\left(F(t)\Big|_a^x\right)$$

$$= \frac{d}{dx}\left(F(x) - F(a)\right)$$

$$\frac{d}{dx}\Big(\underbrace{\int_a^x f(t)\,dt}_{F(t)\big|_a^x}\Big) \quad \frac{d}{dx}\Big(F(t)\Big|_a^x\Big)$$

$$= \frac{d}{dx}\big(F(x) - F(a)\big)$$

$$= F'(x) = f(x) \;//$$

$$F \xrightarrow{\text{derivate}} f$$
$$F \xleftarrow{\text{antiderivate}} f$$

Ex: Find dy/dx

$$y = \int_0^x \sqrt{1+t^2}\,dt$$

$$\frac{dy}{dx} = \frac{d}{dx}\Big(\int_0^x \underbrace{\sqrt{1+t^2}}_{f(t)}\,dt\Big) \overset{FCT1}{=} \underbrace{\sqrt{1+x^2}}_{f(x)}$$

More Generally

$$\frac{d}{dx}\Big(\int_a^{u(x)} f(t)\,dx\Big) \overset{FTC1}{=} f(u(x))\cdot u'(x)$$

Ex) $y = \displaystyle\int_{\sqrt{x}}^0 \sin(t^2)\,dt$

$$\frac{dy}{dt} = \frac{d}{dx}\Big(\int_{\sqrt{x}}^0 \sin(t^2)\,dt\Big) = \frac{d}{dx}\Big(-\int_0^{\overset{u(x)}{\overset{\parallel}{\sqrt{x}}}} \sin(t^2)\,dt\Big)$$

$$= -\frac{d}{dx}\Big(\int_0^{\sqrt{x}} \sin(t^2)\,dt\Big)$$

$$\overset{FTC1}{=} -\sin\big((\sqrt{x})^2\big)\cdot\frac{1}{2\sqrt{x}}$$

Ex) $y = \left(\displaystyle\int_0^x (t^3+1)^{10}\,dt\right)^3$

$$\frac{dy}{dx} = 3\Big(\int_0^x (t^3+1)^{10}\,dt\Big)^2 \cdot \underbrace{\frac{d}{dx}\Big(\int_0^x (t^3+1)^{10}\,dt\Big)}_{\overset{FTC1}{\parallel}\;(x^3+1)^{10}}$$

$$= 3\Big(\int_0^x (t^3+1)^{10}\,dt\Big)^2 \cdot (x^3+1)^{10}$$

More Generally

$u(x)$ ... $f(u(x))\,u'(x)\big)$

More Generally

FTC1 $$\frac{d}{dx}\left(\int_{v(x)}^{u(x)} f(t)\,dt\right) = f(u(x)).u'(x) - f(v(x))\,v'(x)$$

Ex) $y = \int_{2^x}^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt \Rightarrow \frac{dy}{dx} = ?$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_{2^x}^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt\right)$$

$e^{x^2} = u(x)$

2^x → should be constant

$$= \frac{d}{dx}\left(\int_{2^x}^{0} \frac{1}{\sqrt{t}}\,dt + \int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt\right)$$

$$= \frac{d}{dx}\left(\int_{2^x}^{0} \frac{1}{\sqrt{t}}\,dt\right) + \frac{d}{dt}\left(\int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt\right)$$

$$= -\underbrace{\frac{d}{dx}\left(\int_0^{2^x} \frac{1}{\sqrt{t}}\,dt\right)}_{=} + \underbrace{\frac{d}{dt}\left(\int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt\right)}_{=}$$

$2^x = v(x)$, $e^{x^2} = u(x)$

FTC1

$$= -\underbrace{\frac{1}{\sqrt{2^x}}}_{f(v(x))} \cdot \underbrace{2^x . \ln 2}_{v'(x)} + \underbrace{\frac{1}{\sqrt{e^{x^2}}}}_{f(u(x))} \cdot \underbrace{e^{x^2} . 2x}_{u'(x)}$$

Ex: Determine $f(5)$ if f is cont and

$$\int_0^{4x+\sin(\pi x)} f(t)\,dt = x^2 \text{ for all } x.$$

$\frac{d}{dx}$ ↓

$$\underbrace{\frac{d}{dx}\left(\int_0^{4x+\sin(\pi x)} f(t)\,dt\right)}_{\text{FTC1}} = 2x$$

$$f(\underbrace{4x+\sin(\pi x)}_{u(x)}) . \underbrace{(4+\cos(\pi x).\pi)}_{u'(x)} = 2x$$

$x = \frac{3}{2}$ $\quad f\left(\cancel{4}^{2} . \frac{3}{\cancel{2}} + \underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1}\right) . \left(4 + \underbrace{\cos\left(\frac{3\pi}{2}\right)}_{0} . \pi\right) = 2 . \frac{3}{2}$

$$f(5) . 4 = 3 \Rightarrow f(5) = \frac{3}{4}$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$f(x) = F'(x)$

$$\int_a^b F'(x)\,dx = \underbrace{F(x)}_{\text{antiderivative of } f.}\Big|_a^b = F(b) - F(a)$$

Ex: $$\int_{-\pi/3}^{-\pi/4} \underbrace{\left(4\sec^2 t + \frac{\pi}{t^2}\right)}_{f} dt = \overbrace{\left(4 \cdot \tan t + \pi \frac{t^{-1}}{-1}\right)}^{F}\Bigg|_{t=-\pi/3}^{t=-\pi/4}$$

$\frac{\pi}{t^2} = \pi \cdot t^{-2}$

Recall: $x^n \xrightarrow{\text{antiderivative}} \frac{x^{n+1}}{n+1}$

$$= \underbrace{\left(4 \cdot \tan(-\pi/4) - \frac{\pi}{-\pi/4}\right)}_{t=-\pi/4} - \underbrace{\left(4 \cdot \tan(-\pi/3) - \frac{\pi}{-\pi/3}\right)}_{t=-\pi/3}$$

$$= -4 + 4 + 4\sqrt{3} - 3 = 4\sqrt{3} - 3$$

Ex) $$\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2} = \int_0^{1/\sqrt{3}} \frac{dx}{1+(2x)^2} = \frac{1}{2}\arctan(2x)\Big|_0^{1/\sqrt{3}}$$

Recall: $(\arctan x)' = \frac{1}{1+x^2}$

$\left(\frac{1}{2}\arctan 2x\right)' = \frac{1}{\cancel{2}} \frac{1}{1+(2x)^2} \cdot \cancel{2}$

$$= \frac{1}{2}\left[\arctan\left(\frac{2}{\sqrt{3}}\right) - \underbrace{\arctan(0)}_{0}\right]$$

$$= \frac{1}{2}\arctan\left(\frac{2}{\sqrt{3}}\right)$$

Determine $f(5)$ if $\int_0^{f(x)} t^2\,dt = 4x + \sin(\pi x)$

$$\frac{t^3}{3}\Bigg|_0^{f(x)} = 4x + \sin(\pi x)$$

$$\frac{f(x)^3}{3} - \frac{0}{3} = 4x + \sin(\pi x)$$

$$f(x) = \sqrt[3]{12x + 3\sin(\pi x)}$$

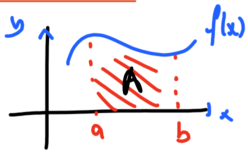
$$f(5) = \sqrt[3]{60 + 3\underbrace{\sin(5\pi)}_{0}} = \sqrt[3]{60}$$

Area

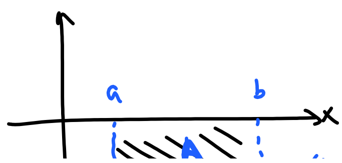
Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

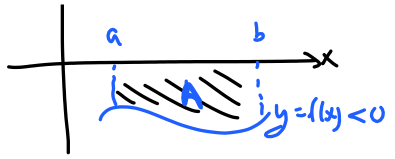
1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.



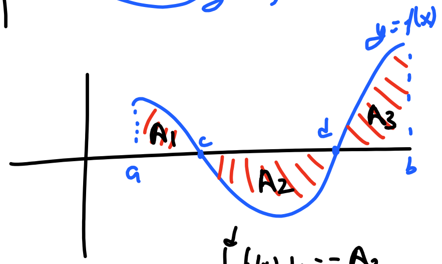
$f(x) > 0 \Rightarrow \int_a^b f(x) = \text{area under the curve} = A$



$\underbrace{\int_a^b f(x)\,dx}_{\text{negative}} = -\underbrace{A}_{\text{positive}}$



$\underbrace{\quad -a \quad}_{\text{negative}}$



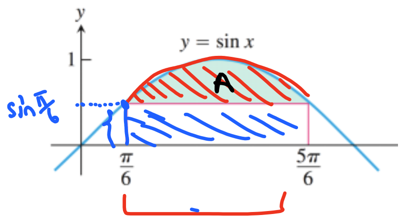
$\int_c^d f(x)\,dx = -A_2$

$\int_a^b f(x)\,dx$ = not area under the curve $= A_1 - A_2 + A_3$

$\neq$

Total area under $y=f(x)$ $= A_1 + A_2 + A_3 = \int_a^b f(x)\,dx + \underbrace{\left|\underbrace{\int_c^d f(x)\,dx}_{\text{negative}}\right|} + \int_d^b f(x)\,dx$

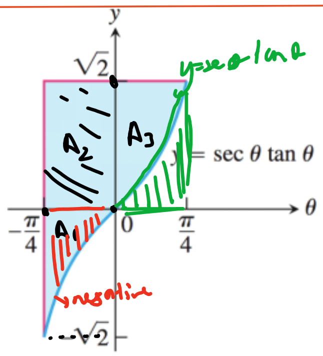
Ex: Find the areas of the shaded regions



Shaded Area $= A = \int_{\pi/6}^{5\pi/6} \sin x\,dx - \underbrace{\sin\frac{\pi}{6}}_{\text{height}} \cdot \underbrace{\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)}_{\text{base}}$

$= -\cos x \Big|_{\pi/6}^{5\pi/6} - \sin\frac{\pi}{6} \cdot \frac{4\pi}{6}$ //

Ex)



$A_1 = \left|\underbrace{\int_{-\pi/4}^{0} \sec\theta \tan\theta\,d\theta}_{\text{negative}}\right| = -\int_{-\pi/4}^{0} \underbrace{\sec\theta\tan\theta}\,d\theta = -\sec\theta \Big|_{-\pi/4}^{0}$

$A_2 = \frac{\pi}{4} \cdot \sqrt{2}$

$A_3 = \frac{\pi}{4} \cdot \sqrt{2} - \underbrace{\int_0^{\pi/4} \sec\theta \tan\theta\,d\theta}_{\sec\theta \Big|_0^{\pi/4}}$

Recall

$(\sec\theta)' = \sec\theta \tan\theta$

$A_1 + A_2 + A_3 = ?$

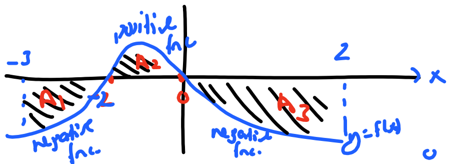
$A_1 + A_2 + A_3 = \underbrace{-\sec\theta \Big|_{-\pi/4}^{0}}_{A_1} + \underbrace{\frac{\pi}{4} \cdot \sqrt{2}}_{A_2} + \underbrace{\frac{\pi}{4} \cdot \sqrt{2} - \sec\theta \Big|_0^{\pi/4}}_{A_3}$

Ex: find the total area between the region and the x-axis.

$y = \underbrace{-x^2 - 2x}_{f(x)}, \quad -3 \le x \le 2$

$y = -x(x+2) < \begin{matrix} x=0 \\ x=-2 \end{matrix}$

$\begin{array}{c|ccccc} & & -2 & & 0 & \\ \hline y & - & 0 & + & 0 & - \end{array}$



Total area $= \underbrace{\left|\underbrace{\int_{-3}^{-2} -x^2 - 2x\,dx}_{\text{negative}}\right|}_{A_1} + \underbrace{\int_{-2}^{0} -x^2 - 2x\,dx}_{= A_2} + \underbrace{\left|\underbrace{\int_0^2 -x^2 - 2x\,dx}_{\text{negative}}\right|}_{A_3}$

$= \left|\left(-\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_{-3}^{-2}\right| + \left(-\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_{-2}^{0} + \left|\left(-\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_{0}^{2}\right|$

$\ldots \left| 0 - \left(\frac{8}{3} - 4\right) \right| + \left| -\frac{8}{3} - 4 - 0 \right|$

$= \left|\left(-\frac{x^3}{3} - \frac{2x^2}{2}\right)\Big|_{-3}^{-2}\right| + \left(-\frac{x^3}{3} - \frac{2x^2}{2}\right)\Big|_{-2}^{0} + \left|\left(-\frac{x^3}{3} - \frac{2x^2}{2}\right)\Big|_{0}^{}\right|$

$= \left|\left(-\frac{8}{3} - 4\right) - \left(\frac{27}{3} - 9\right)\right| + \left|0 - \left(\frac{8}{3} - 4\right)\right| + \left|\frac{-8}{3} - 4 - 0\right|$

$= \frac{28}{3}$ //

5.5 Indefinite Integrals and the Substitution Method

Definite Integral: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \rightsquigarrow$ number!

antiderivative = integral

derivative

Indefinite Integral: $\int f(x)\,dx = F(x) + C \leftarrow$ arbitrary constant.

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

Substitution: $u = g(x)$, $du = g'(x)dx$

$$\int f(\underbrace{g(x)}_{u})\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du. = F(u) + C$$

Ex: $\int x \sin(\underbrace{2x^2}_{u})\,dx$ $\quad$ substitution $u = 2x^2$, $du = 4x\,dx$, $x\,dx = \frac{du}{4}$

$\int \sin(u)\frac{du}{4} = \frac{1}{4}\int \sin u\,du = \frac{1}{4}(-\cos u) + C \overset{\text{back subs.}}{=} -\frac{1}{4}\cos(2x^2) + C$

Ex) $\int \sec \underbrace{2t}_{u} \tan \underbrace{2t}_{u}\,\underbrace{dt}_{\frac{du}{2}}$, $\quad$ Subs: $u = 2t$, $du = 2\,dt$

$= \int \sec u \tan u \frac{du}{2} = \frac{1}{2}\int \sec u \tan u\,du = \frac{1}{2}\sec u + C$

$= \frac{1}{2}\sec(2t) + C$

Ex) $\int \left(\underbrace{1 - \cos\frac{t}{2}}_{u}\right)^2 \underbrace{\sin\frac{t}{2}\,dt}_{2du} = \int u^2\, 2\,du$

Subs) $u = 1 - \cos\frac{t}{2}$

$du = \sin\frac{t}{2}\cdot\frac{1}{2}\,dt$

$= 2\int u^2\,du$

$= 2\cdot\frac{u^3}{3} + C$

$\overset{\text{back subs.}}{=} 2\cdot\frac{\left(1-\cos\frac{t}{2}\right)^3}{3} + C$

Ex) $\int \frac{(\overbrace{1+\sqrt{x}}^{u})^{1/3}}{\sqrt{x}}\,dx = \int u^{1/3}\,2\,du$

Subs. $u = 1 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int u^{1/3}\,du$

$= 2\cdot\frac{u^{1/3+1}}{1/3+1} + C$

Subs: $u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$$= 2\int u^{1/3}\,du = 2\cdot\frac{u^{1/3+1}}{1/3+1} + C = 2\,\frac{(1+\sqrt{x})^{4/3}}{4/3} + C$$

Ex) $$\int \frac{1}{\sqrt{\underbrace{5s+4}_{u}}}\,ds = \int \frac{1}{\sqrt{u}}\,\frac{du}{5}$$

$u = 5s+4$

$du = 5ds$

$$= \frac{1}{5}\int \frac{du}{\sqrt{u}} = \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5}\,\frac{u^{-1/2+1}}{-1/2+1} + C = \frac{1}{5}\,\frac{(5s+4)^{1/2}}{1/2} + C = \frac{2}{5}\sqrt{5s+4} + C$$

Ex) $$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = \int \frac{2\,du}{u^2} = 2\int u^{-2}\,du$$

Subs) $u = \sin\sqrt{\theta}$

$du = \cos\sqrt{\theta}\cdot\frac{1}{2\sqrt{\theta}}\cdot d\theta$

$$= 2\,\frac{u^{-1}}{-1} + C = -\frac{2}{u} + C = -\frac{2}{\sin\sqrt{\theta}} + C$$

$$\int \sqrt{\frac{x-1}{x^5}}\,dx$$

The Integrals of $\sin^2 x$ and $\cos^2 x$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Recall)

$\int \sin kx = -\frac{\cos kx}{k} + C$

$\int \cos kx = -\frac{\sin kx}{k} + C$

- $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$
- $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int 1 + \cos 2x\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

5.6 Substitution and Area Between Curves

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du.$$

boundaries for "x"

Substitution $u = g(x)$, $du = g'(x)\,dx$

are boundaries for "u"

In 16

Substitution
$u = g(x)$
$du = g'(x)\,dx$

for "u"

Ex: $\displaystyle\int_{2}^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} \frac{du}{\sqrt{u}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} u^{-1/2}\,du = \cancel{\frac{1}{2}}\,\frac{u^{1/2}}{\cancel{1/2}}\Big|_{\ln 2}^{\ln 16} + \cancel{C}$

$= (\ln 16)^{1/2} - (\ln 2)^{1/2}$

$= \sqrt{\ln 16} - \sqrt{\ln 2}$ //

Substitution
$u = \ln x$ — $x = 2 \Rightarrow u = \ln 2$; $x = 16 \Rightarrow u = \ln 16$ — boundaries for "u"
$du = \frac{dx}{x}$

Definite Integrals of Symmetric Functions

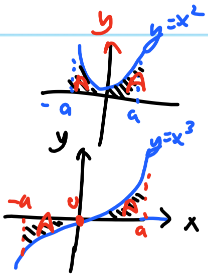
THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

(b) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

• f is even : $f(x) = f(-x)$
↳ symmetric w.r.t "y" axis

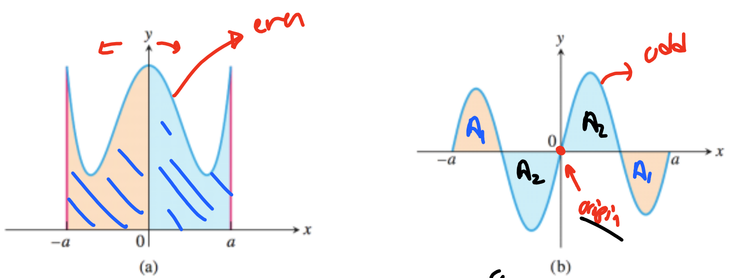
• f is odd : $f(x) = -f(x)$
↳ symmetric w.r.t "origin"



$\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
$= 2.A$

$\displaystyle\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{-A} + \underbrace{\int_{0}^{a} f(x)\,dx}_{A}$

$= -A + A = 0$ //



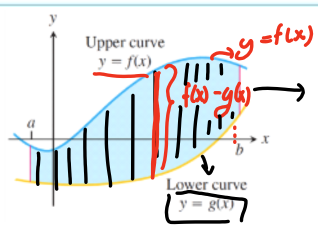
$\displaystyle\int_{-a}^{a} f(x)\,dx = 2.\int_{0}^{a} f(x)\,dx$

$\displaystyle\int_{-a}^{a} f(x) = \cancel{A_1} - \cancel{A_2} + \cancel{A_2} - \cancel{A_1}$
$= 0$ //

Areas Between Curves

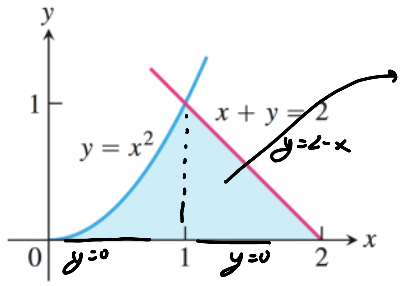
DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_{a}^{b} [f(x) - g(x)]\,dx.$$



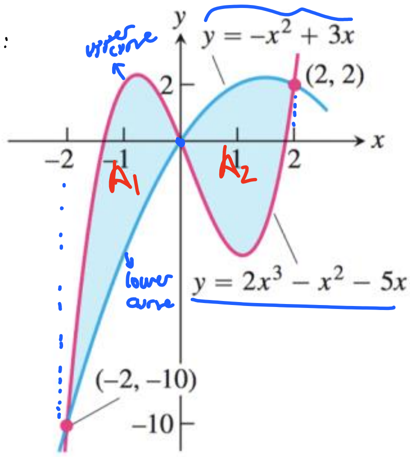
area between curves $\displaystyle = \int_{a}^{b} f(x) - g(x)\,dx$ — w.r.t "x"

Ex:



Shaded Area $= \int_0^1 (x^2-0)\,dx + \int_1^2 (2-x-0)\,dx$

Ex:

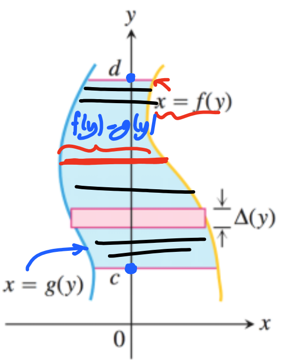


Area between curves $= \overbrace{\int_{-2}^{0} (\overbrace{2x^3-x^2-5x}^{\text{upper}}) - (\overbrace{-x^2+3x}^{\text{lower}})\,dx}^{A_1}$

$+$

$\underbrace{\int_{0}^{2} (\overbrace{-x^2+3x}^{\text{upper}}) - (\overbrace{2x^3-x^2-5x}^{\text{lower}})\,dx}_{A_2}$

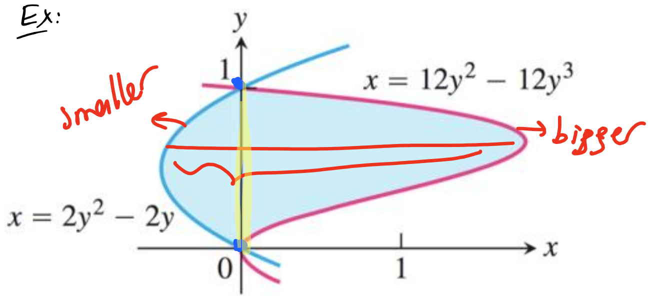
Integration with Respect to y



w.r.t "y"

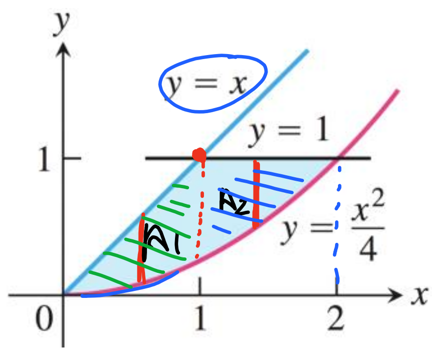
Area between curves $= \int_c^d \underset{\text{bigger}}{f(y)} - \underset{\text{smaller}}{g(y)}\;dy$

Ex:

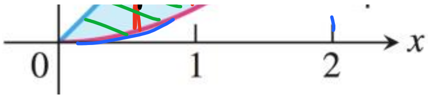


$\int_0^1 (\underbrace{12y^2-12y^3}_{\text{bigger}}) - (\underbrace{2y^2-2y}_{\text{smaller}})\,dy$

Ex:



1 bigger smaller. 2 bigger smaller

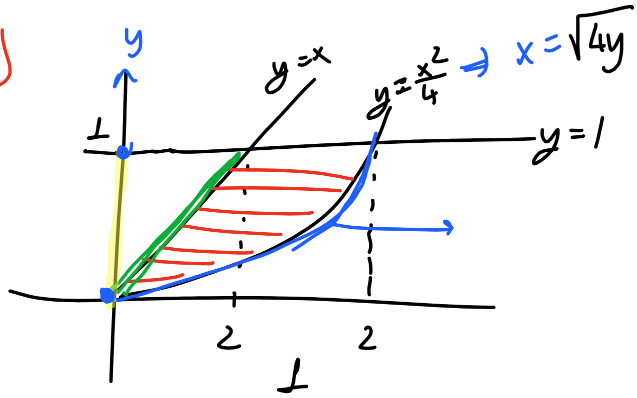


1 way)

bigger smaller. bigger smaller

Total Area $= A_1 + A_2 = \int_0^1 \left(x - \frac{x^2}{4}\right) dx + \int_1^2 \left(1 - \frac{x^2}{4}\right) dx$

2 way)



Total Area $= \int_0^1 \sqrt{4y} - y \; dy$

$= \frac{1}{4} \cdot \frac{(4y)^{3/2}}{3/2} \Big|_0^1 - \frac{y^2}{2} \Big|_0^1$

$= \left(\frac{1}{6} \cdot 8 - 0\right) - \left(\frac{1}{2} - 0\right) = \frac{5}{6}$

$\int \sqrt{4y}\, dy = \frac{1}{4}\int u^{1/2}\, du = \frac{1}{4} \frac{u^{3/2}}{3/2}$

$\frac{du}{4}$

$u = 4y$

$du = 4dy$

Ex: Find the areas of the regions enclosed by the curves

- $y^2 - 4x = 4$ and $4x - y = 16$
- $x = y^3 - y^2$ and $x = 2y$
- $y = \sqrt{|x|}$ and $5y = x + 6$
- $y = 8\cos x$ and $y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$