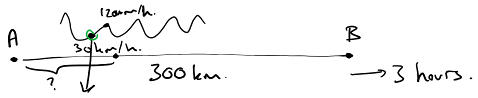
Lecture

2

Limits and Continuity

2.1 Rates of Change and Tangents to Curves

Average and Instantenous Rate of Change



instantaneous speed.

at a moment

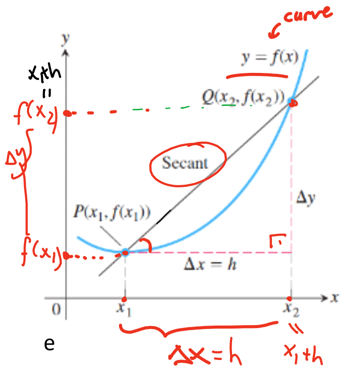
$\frac{\Delta y}{\Delta t} = \frac{300km}{3h} = 100km/h.$

Average speed.

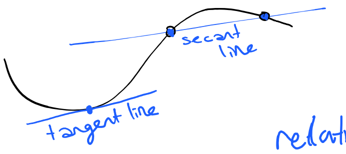
in an interval.

DEFINITION The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$



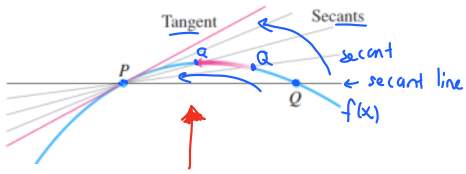
e



relation between secant and tangent

as $Q \to P$

secant lines $\xrightarrow{Q \to P}$ tangent line



limit ⟶ slope of tangent line?

secant lines: slope of secant line $= m_{PQ}$

tangent line: slope of tangent $= m_t = \lim\limits_{Q \to P} m_{PQ}$

$$\boxed{\text{avg. rate of change of } f = \frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h}}$$

avg. rate of change of f = slope of secant

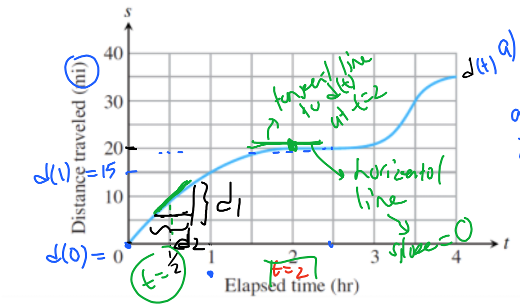
instantenous rate of change of f $= \lim\limits_{Q \to P} \frac{f(x_1+h) - f(x_1)}{h}$

instantenous rate of change of f = slope of tangent at that moment

Ex: The accompanying graph shows the total distance s traveled by a bicyclist after t hours.

15 − 0   15 mil.

Ex: The accompanying graph shows the total distance s traveled by a bicyclist after t hours.



$$S_{[0,1]} = \frac{d(1) - d(0)}{1 - 0} = \frac{\Delta d}{\Delta t} = \frac{15 - 0}{1 - 0} = 15 \text{ mi/hr}$$

average speed over $[0,1]$

$$S_{[1,2.5]} = \frac{d(2.5) - d(1)}{2.5 - 1} = \frac{20 - 15}{1.5} = \frac{5}{1.5} \text{ mi/hr}$$

average speed over $[1, 2.5]$

a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.

b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.

b) $t = 2$: instantaneous speed at $t=2$ = slope of tangent line to $d(t)$ when $t=2$ = 0 (horizontal line)

$S_2 = 0$ mi/h

$t = \frac{1}{2}$: instantaneous speed $t = 1/2$ = slope of tangent line to $d(t)$ at $t = \frac{1}{2}$ $= \frac{d_1}{d_2}$ → estimate by looking at the graph!

2.2 Limit of a Function and Limit Laws

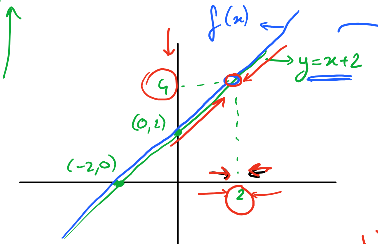
As x goes to 2 behavior of $f(x)=(x^2-4)/(x-2)$

x	f(x)
1	3
1.5	3.5
1.8	3.8
1.9	3.9
1.91	3.91
1.95	3.95
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999
2	#DIV/0!
2.0001	4.0001
2.001	4.001
2.01	4.01
2.1	4.1
2.5	4.5
3	5
4	6
8	10

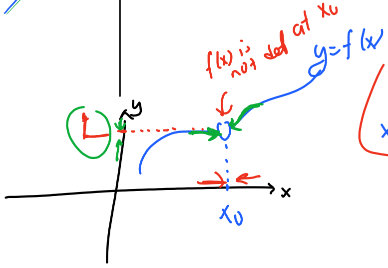
#DIV/0! → undefined.

$$f(x) = \frac{x^2-4}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x+2, \quad x \neq 2$$

at $x=2$ ⇝ leads a division by 0 : $f(2) = \frac{0}{0}$ = undef



$$\lim_{x \to 2} f(x) = 4$$



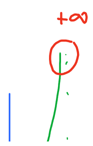
$$\boxed{\lim_{x \to x_0} f(x) = L}$$

Definition of a limit : if the value of $f(x)$ are close to L (approaches to / goes to) whenever x is close to x_0 (approaches / goes):

$$\boxed{\lim_{x \to x_0} f(x) = L}$$

As x goes to 2 behavior of $f(x)=(x-4)/(x-2)$

x	f(x)
1	3
1.5	5
1.8	11

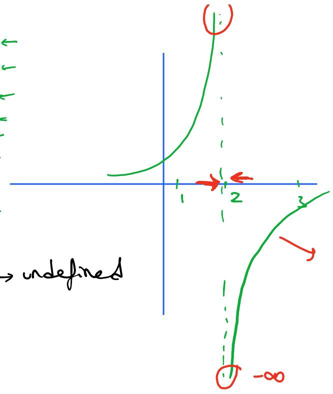


As x goes to 2
behavior of
f(x)=(x -4) / (x-2)

$$f(x) = \frac{x-4}{x-2}$$

x	f(x)
1	3
1.5	5
1.8	11
1.9	21
1.91	23.22222222
1.95	41
1.99	201
1.999	2001
1.9999	20001
1.99999	200001
2	#DIV/0!
2.0001	-19999
2.001	-1999
2.01	-199
2.1	-19
2.5	-3
3	-1
4	0
8	0.666666667

#DIV/0! → undefined



$f(x) = \frac{x-4}{x-2}$

1 → 2 ← 3

$f(2) = \frac{-2}{0}$ = undefined

x=2

$\lim_{x \to 2} f(x)$ = does not exist.

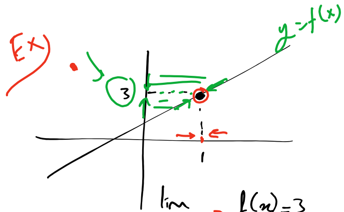
fact. $\frac{\text{finite number}}{\text{zero}}$ = undef.

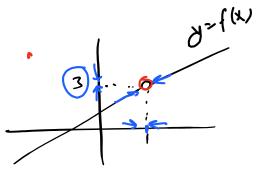
$f(2)$ = undefined

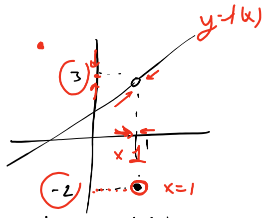
$\lim_{x \to 2} \frac{x-4}{x-2}$ = DNE / 0

→ 2 ←
1,9999999, 2,001

Ex)







$\lim_{x \to 1} f(x) = 3$

$f(1) = 3$

$\lim_{x \to 1} f(x) = 3$

$f(1)$ = undef.

x=1

$\lim_{x \to 1} f(x) = 3$ ≠

$f(1) = -2$

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$ $= \lim f(x) + \lim g(x)$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$ $= \lim f(x) - \lim g(x)$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$ $k \in \mathbb{R}$ $= k.\lim_{x \to c} f(x)$, L
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$ $\lim g(x) \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

2.4 One-Sided Limits

left side limit (hand) approaches from left

right side limit (hand) approaches from right left side limit (hand)

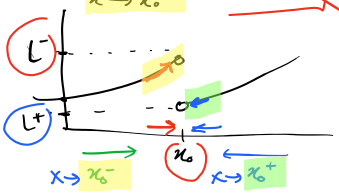
approaches from left

$$\boxed{\lim_{x \to x_0^-} f(x) = L^-}$$

right side limit (hand)

approaches from right

$$\lim_{x \to x_0^+} f(x) = L^+$$



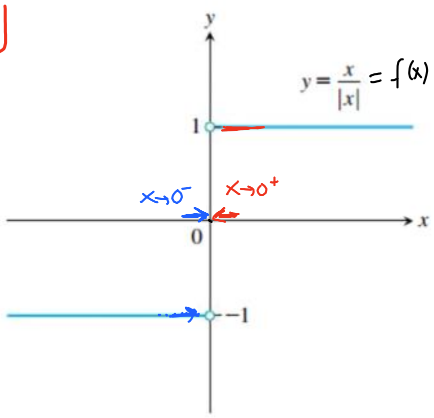
if they are the same $\boxed{L^- = L^+}$ then limit exists and $\boxed{\lim_{x \to x_0} f(x) = L = L^- = L^+}$

if they are not same $L^- \neq L^+$

$\Downarrow$

$\lim_{x \to x_0} f(x) = DNE$

Ex



$y = \frac{x}{|x|} = f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

$x<0 \to |x| = -x \rightsquigarrow f(x) = \frac{x}{-x} = -1$

$x>0 \to |x| = x \rightsquigarrow f(x) = \frac{x}{x} = 1$

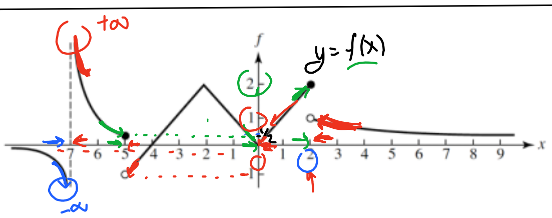
$\lim_{x \to 0^-} f(x) = -1 \neq \lim_{x \to 0^+} f(x) = 1$

$\Downarrow$

$\lim_{x \to 0} f(x) = DNE$!

FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Ex



$x = -7$: $\lim_{x \to -7^-} f(x) = -\infty$, $\lim_{x \to -7^+} f(x) = +\infty$ $\Big\rangle$ $\lim_{x \to -7} f(x) = DNE$

$x = -5$: $\lim_{x \to -5^-} f(x) = \frac{1}{2}$ $\neq$ $\lim_{x \to -5^+} f(x) = -1$ $\Big\rangle$ $\lim_{x \to -5} f(x) = DNE$

$x = 0$: $\lim_{x \to 0^-} f(x) = 0$ $\|$ $\lim_{x \to 0^+} f(x) = 0$ $\to$ $\lim_{x \to 0} f(x) = 0$

$x = 2$: $\lim_{x \to 2^-} f(x) = 2$ $\neq$ $\to$ $\lim f(x) = DNE$

$x=2$: $\lim_{x\to 2^-} f(x) = 2$ $\neq$ → $\lim_{x\to 2} f(x) = DNE$

$\lim_{x\to 2^+} f(x) = 1$

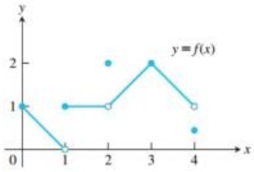


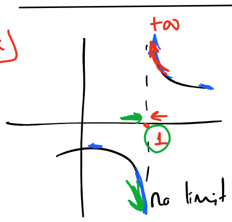
FIGURE 2.27 Graph of the function in Example 2.

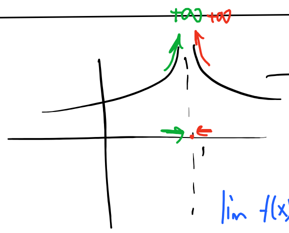
EXAMPLE 2 For the function graphed in Figure 2.27,

At $x = 0$:	$\lim_{x\to 0^-} f(x)$ does not exist,	f is not defined to the left of $x = 0$.
	$\lim_{x\to 0^+} f(x) = 1$,	f has a right-hand limit at $x = 0$.
	$\lim_{x\to 0} f(x) = 1$.	f has a limit at domain endpoint $x = 0$.
At $x = 1$:	$\lim_{x\to 1^-} f(x) = 0$,	Even though $f(1) = 1$.
	$\lim_{x\to 1^+} f(x) = 1$,	
	$\lim_{x\to 1} f(x)$ does not exist.	Right- and left-hand limits are not equal.
At $x = 2$:	$\lim_{x\to 2^-} f(x) = 1$,	
	$\lim_{x\to 2^+} f(x) = 1$,	
	$\lim_{x\to 2} f(x) = 1$.	Even though $f(2) = 2$.
At $x = 3$:	$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.	
At $x = 4$:	$\lim_{x\to 4^-} f(x) = 1$,	Even though $f(4) \neq 1$.
	$\lim_{x\to 4^+} f(x)$ does not exist,	f is not defined to the right of $x = 4$.
	$\lim_{x\to 4} f(x) = 1$.	f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$.

Ex)





→ $\lim_{x\to 1} f(x) = \infty$

$\lim_{x\to 1^-} f(x) = -\infty$ } $\lim_{x\to 1} f(x) = DNE$

$\lim_{x\to 1^+} f(x) = +\infty$

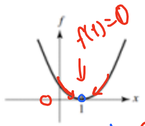
$\lim_{x\to 1^-} f(x) = +\infty$ } $\lim_{x\to 1} f(x) = \infty$ ← giving an extra information

↳ lim doesn't exist

$\lim_{x\to 1^+} f(x) = +\infty$

Ex)

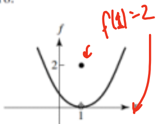
17.



$f(1)=0$

lim exist $= 0$

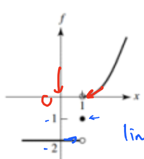
18.



$f(1)=2$

lim exist $= 0$

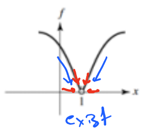
19.



limit does not exist

$\lim_{x\to 1^-} f(x) = -2 \neq \lim_{x\to 1^+} f(x) = 0$

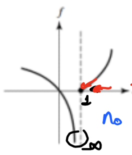
20.



exist

$\lim_{x\to 1} f(x) = 0$

21.

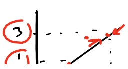


$\lim_{x\to 1^+} f(x) = 0$

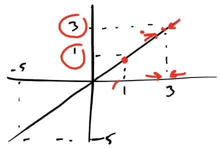
no limit $\not\!\!\times$ ⇒ $\lim_{x\to 1} f(x) = DNE$

$\lim_{x\to 1^-} f(x) = -\infty$

Ex: Identity function $\boxed{f(x) = x}$



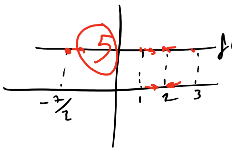
$\boxed{\lim_{x\to a} x = a}$



$$\boxed{\lim_{x\to a} x = a}$$

$$\lim_{x\to 3} x = 3$$

Ex: Constant function $f(x) = c \rightarrow \lim_{x\to a} c = c$



$$\lim_{x\to a} 5 = 5$$

Remark

a)

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.

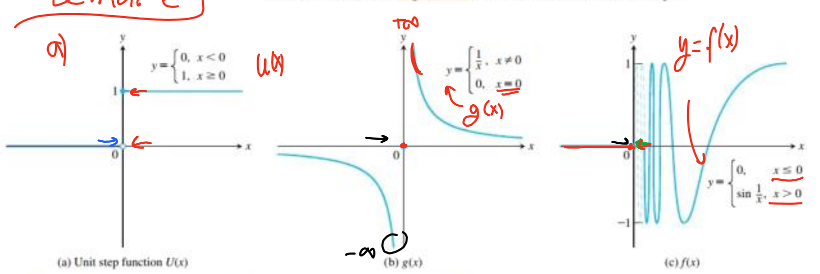


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$ $\rightarrow$ $\lim_{x\to 0^-} U(x) = 0 \neq \lim_{x\to 0^+} U(x) = 1 \Rightarrow \lim_{x\to 0} U(x) = DNE$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\longrightarrow$ $\lim_{x\to 0^-} g(x) = -\infty$ $\quad \lim_{x\to 0^+} g(x) = \infty$

$\lim_{x\to 0} g(x) = DNE$

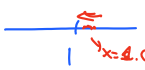
(c) $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$ $\longrightarrow$ $\lim_{x\to 0^-} f(x) = 0$ $\quad \lim_{x\to 0^+} f(x) = DNE$

$\lim_{x\to 0} f(x) = DNE$

Ex)

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} \qquad\qquad \lim_{x\to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|}$$

positive $\qquad$ negative



$x = 1.0001 \rightsquigarrow |x-1| = x-1$

$x = 0.00009$

$|x-1| = -(x-1)$

$$= \lim_{x\to 1^+} \frac{\sqrt{2x}\,\cancel{(x-1)}}{\cancel{(x-1)}} \qquad\qquad = \lim_{x\to 1^-} \frac{\sqrt{2x}\,\cancel{(x-1)}}{-\cancel{(x-1)}}$$

$$= \lim_{x\to 1^+} \sqrt{2x} = \sqrt{2} \qquad\qquad = \lim_{x\to 1^-} -\sqrt{2x} = -\sqrt{2}$$

$x \to 1^+$ $\qquad$ $x \to 1^-$

They are different

$$\lim_{x\to 1} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \text{DNE}$$

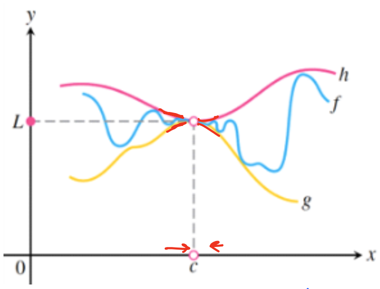
Sandwich Theorem

assumption

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.



(Sequeze) Sandwich Thm

assumtion: $g(x) \le f(x) \le h(x)$

and

$x\to c$ $\qquad$ $x\to c$

L

$\Downarrow$

conclusion: $\boxed{\lim_{x\to c} f(x) = L}$

Ex: If $\boxed{\underbrace{2 - \frac{x^2}{5}}_{g(x)} \le f(x) \le \underbrace{2 + \frac{x^2}{3}}_{h(x)}}$ for all $x \neq 0$. $\rightarrow$ inequality

Then find $\lim_{x\to 0} f(x)$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \underbrace{2 - \frac{x^3}{5}}_{\text{poly}} = 2$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \underbrace{2 + \frac{x^2}{3}}_{\text{poly}} = 2$$

$\Vert \xRightarrow{\text{Sandwich Thm}} \lim_{x\to 0} f(x) = 2$ //

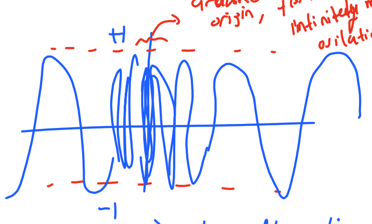
Ex: If $2 - x^2 \le g(x) \le 2\cos x$ for all x, then find $\lim_{x\to 0} g(x)$.

$$\lim_{x\to 0} 2 - x^2 = 2$$

$$\lim_{x\to 0} 2\cos x = 2\underbrace{\cos 0}_{1} = 2$$

$\xrightarrow[\text{Thm}]{\text{Sandwich}}$ $\lim_{x\to 0} g(x) = 2$

Ex) $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

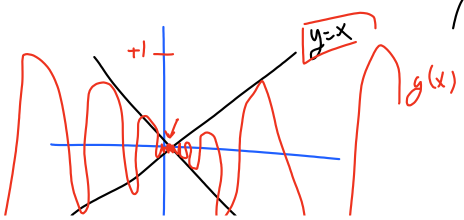


around origin, f(x) makes infinitely many osilation so that there is no unique number to which f(x) is aproaching

$\Downarrow$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\frac{1}{x} = \text{DNE}$

$g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$



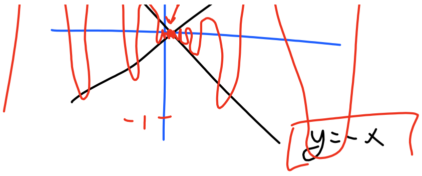
1st way) from the graph:

$\lim_{x\to 0} g(x) = 0$

Recall) $-1 \le \sin\theta \le 1$

2nd way) Sandwich Thm:

$\sin\frac{1}{x} \le 1$



2nd way) Sandwich Thm:

$-1 \le \sin\frac{1}{x} \le 1$

$\boxed{x>0}$: $-x \le x\sin\frac{1}{x} \le x$

$x\to 0$: both sides $\to 0$

$\Downarrow$ By Sandwich Thm

$\lim_{x\to 0^+} x\sin\frac{1}{x} = 0$

$\boxed{x<0}$: $-x \ge x\sin\frac{1}{x} \ge x$

$x\to 0$: both sides $\to 0$

$\Downarrow$ Sandwich Thm

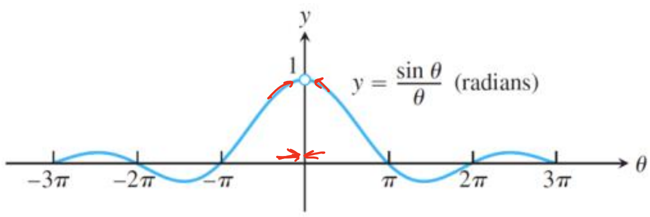
$\lim_{x\to 0^-} x\sin\frac{1}{x} = 0$

by Sandw.

$\boxed{\lim_{x\to 0} x\sin\frac{1}{x} = 0}$

Fact:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1} \Rightarrow \boxed{\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1}$$



Ex: $\lim_{y\to 0}\dfrac{\frac{3}{4}\sin 3y}{\frac{3}{4}4y} \sim \dfrac{0}{0} \to$ indet. form.

$$= \lim_{y\to 0}\frac{\frac{3}{\cancel{4}}\sin 3y}{\frac{3}{\cancel{4}}\cdot\cancel{4}y} = \lim_{y\to 0}\frac{3}{4}\underbrace{\left(\frac{\sin 3y}{3y}\right)}_{\to 1} = \frac{3}{4}$$

Ex: $\dfrac{0}{0}$ $\lim_{t\to 0}\dfrac{2t}{\tan t} = \lim_{t\to 0}\dfrac{2t}{\frac{\sin t}{\cos t}} = \lim_{t\to 0}\dfrac{\cos t\cdot 2t}{\sin t} = \lim_{t\to 0} 2\cos t = 2\cdot\underbrace{\cos 0}_{1}$

$= 2$ //